Network Utility Optimization

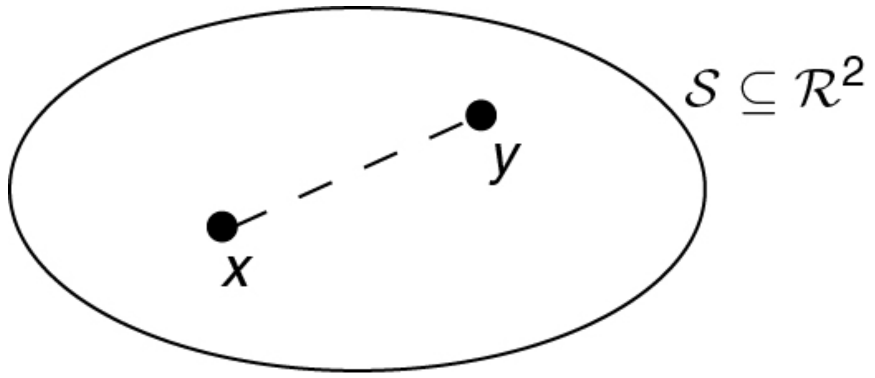
CMPS 4750/6750: Computer Networks

Outline

- Convex Optimization (SY 2.1)
- Network Utility Maximization (SY 2.2)
- Utility Functions and Fairness (SY 2.2.1)

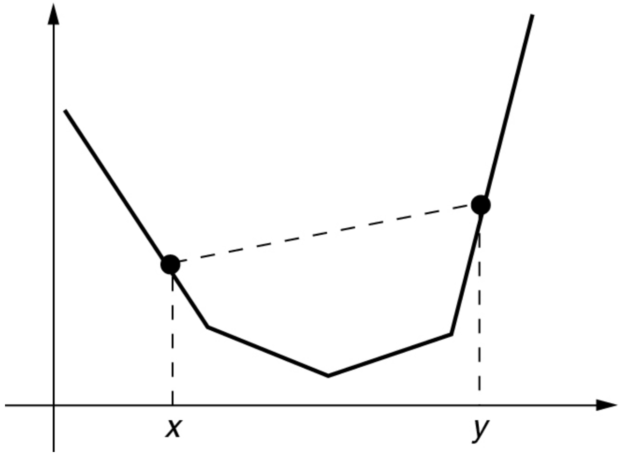
Convex Sets

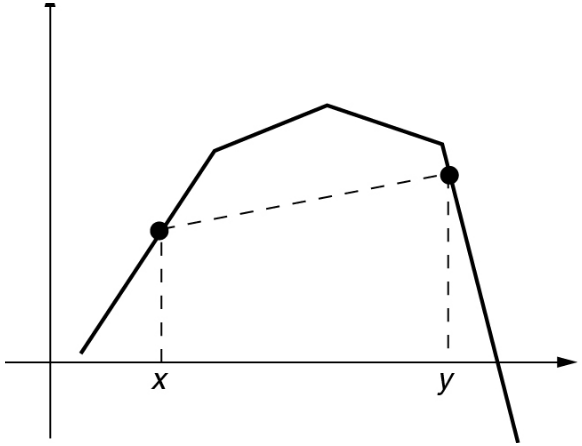
- Convex Set $S \subseteq R^n$: $if\ x \in S$ and $y \in S$, then $\alpha x + (1-\alpha)y \in S$ for $\alpha \in [0,1]$



Convex and Concave Functions

- Convex function $f(x)$: $S \to R$: $S \subseteq R^n$ is a convex set and for any $x, y \in S$ and $\alpha \in [0, 1]$: $f\,(\alpha x + (1-\alpha)y) \;\leq\; \alpha f(x) + (1-\alpha)f(y)$



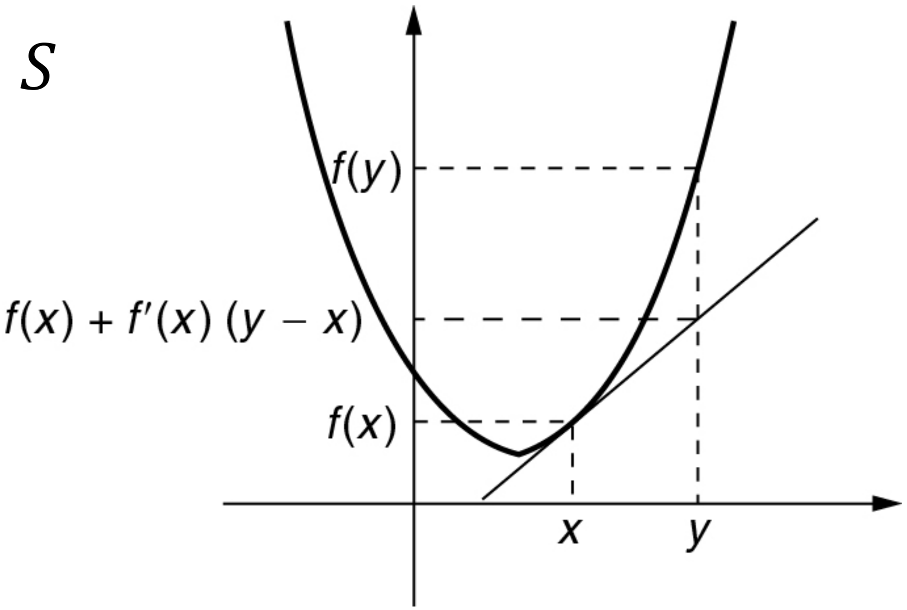


- A function $f(x)$: $S \to R$ is concave if $-f$ is convex

Convex Functions: First-Order Condition

- Let $f(x)\colon\ S \to R$ be differentiable and $S \subseteq R^n$ be convex. f is convex if and only if

$$f\ (y) \geq f(x) + \nabla f(x)(y - x), \qquad \forall x, y \in S$$



Unconstrained Convex Optimization

$$\max_{x \in S} f(x)$$

Fact: If f is concave and differentiable and S is convex, then x^* is a maximizer if and only if $\nabla f(x^*)(x - x^*) \leq 0$ for $x \in S$.

Network Utility Maximization

- $\mathcal{L}$: set of links, $\mathcal{S}$: set of sources (users)
- Each source has a fixed route (a collection of links)
- $U_r(x_r)$: utility of source r when transmitting at rate x_r
  - non-decreasing: $U_r(x) \geq U_r(y)\ if\ x \geq y$
  - concave: $U_r(\alpha x + (1-\alpha)y) \geq \alpha U_r(x) + (1-\alpha)U_r(y)$

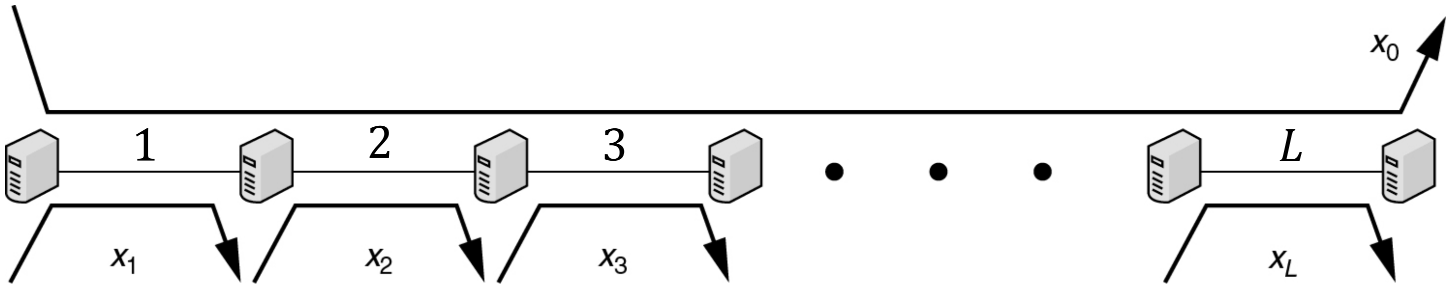
Network Utility Maximization

- Network Utility Maximization (NUM)

$$\max_{x_r} \sum_{r \in S} U_r(x_r)$$

$$\left.\begin{aligned} s.t. \quad & \sum_{r:l \in r} x_r \le c_l, \ \forall\, l \in \mathcal{L}, \\ & x_r \ge 0, \qquad \forall\, r \in \mathcal{S}. \end{aligned}\right\} \quad x \in \mathcal{D}$$

Example



$$\max_{x_r} \sum_{r=0}^{L} \log x_r \qquad s.t. \quad x_0 + x_l \le 1, \qquad \forall\, l = 1, \dots, L,$$
$$x_r \ge 0, \qquad \forall\, r = 0, \dots, L.$$

Lagrangian multipliers ≥ 0

Lagrangian: $L(x, p) = \sum_{r=0}^{L} \log x_r - \sum_{l=1}^{L} p_l (x_0 + x_l - 1)$

KKT conditions: x is a global maximizer iff there exists p such that

(1) $\frac{\partial L}{\partial x_r} = 0$ for each r; ⇒ $x_0 = \frac{1}{\sum_{l=1}^{L} p_l}$, $x_r = \frac{1}{p_r}, \forall r \ge 1$

(2) $p_l (x_0 + x_l - 1) = 0$ for each l. ⇒ $p_l = \frac{L+1}{L}, \forall l \ge 1$

} $x_0 = \frac{1}{L+1}$, $x_r = \frac{L}{L+1}, \forall r \ge 1$

Utility Functions and Fairness

- A utility function can be interpreted as
  - an inherent utility associate with each user
  - imposing a notion of fair resource allocation

- Proportional fairness

An allocation x^* is proportional fair if $\sum_{r \in \mathcal{S}} \frac{x_r - x_r^*}{x_r^*} \leq 0$ for any feasible allocation x

$\Leftrightarrow$ x^* is the optimal solution to $\max_{x \in \mathcal{D}} \sum_{r \in S} \log x_r$ (from the optimality condition)

Utility Functions and Fairness

- Max-min fairness: an allocation x^* is max-min fair if for any feasible x, if $x_s > x_s^*$, there is u such that $x_u < x_u^* \le x_s^*$

$$\Rightarrow \min_r x_r^* \ \ge \min_r x_r$$

- α-fairness: x^* is the optimal solution to $\max_{x \in D} \sum_{r \in S} U_r(x_r)$ where $U_r(x_r) = \frac{x_r^{1-\alpha}}{1-\alpha}$ for some $\alpha > 0$
    - $\alpha \to 1 \ \Rightarrow$ proportional fairness (because $\lim_{\alpha \to 1} \frac{x_r^{1-\alpha} - 1}{1-\alpha} = \log x_r$)
    - $\alpha \to \infty \Rightarrow$ max-min fairness