CMPS 6610/4610 – Fall 2016

Divide-and-Conquer

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Slides courtesy of Charles Leiserson
with changes and additions by Carola Wenk

The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems of sizes that are fractions of the original problem size.
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.

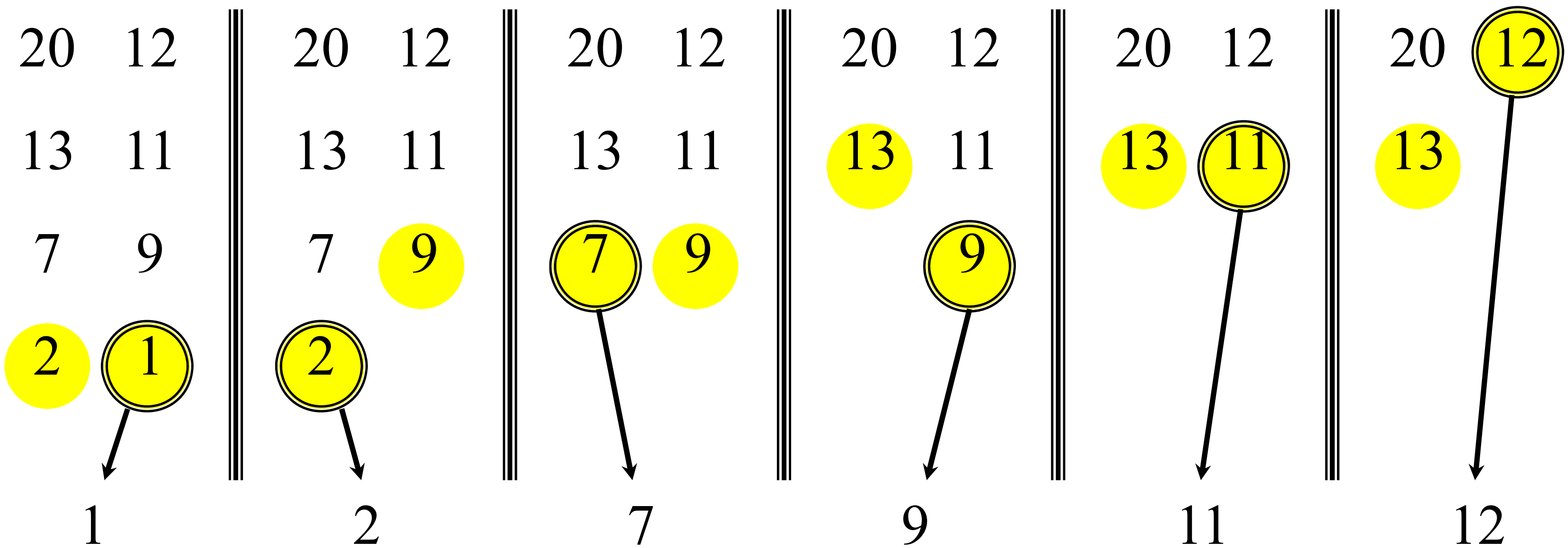
Merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays of size $n/2$
3. ***Combine:*** Linear-time key subroutine **MERGE**

MERGE-SORT ($A[0 \,..\, n\text{-}1]$)

1. If $n = 1$, done.
2. **MERGE-SORT** ($A[\,0 \,..\, \lceil n/2 \rceil\text{-}1]$)
3. **MERGE-SORT** ($A[\,\lceil n/2 \rceil \,..\, n\text{-}1\,]$)
4. "***Merge***" the 2 sorted lists.

Merging two sorted arrays



Time $dn \in \Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

$T(n)$	MERGE-SORT ($A[0\,.\,.\,n\text{-}1]$)
d_0	1. If $n = 1$, done.
$T(n/2)$	2. MERGE-SORT ($A[\,0\,.\,.\lceil n/2\rceil+1]$)
$T(n/2)$	3. MERGE-SORT ($A[\,\lceil n/2\rceil\,.\,.\,n\text{-}1\,]$)
dn	4. "***Merge***" the 2 sorted lists.

Sloppiness: Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.

Recurrence for merge sort

$$T(n) = \begin{cases} d_0 \text{ if } n = 1; \\ 2T(n/2) + dn \text{ if } n > 1. \end{cases}$$

- But what does $T(n)$ solve to? I.e., is it $O(n)$ or $O(n^2)$ or $O(n^3)$ or …?

Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

$$T(n)$$

Recursion tree

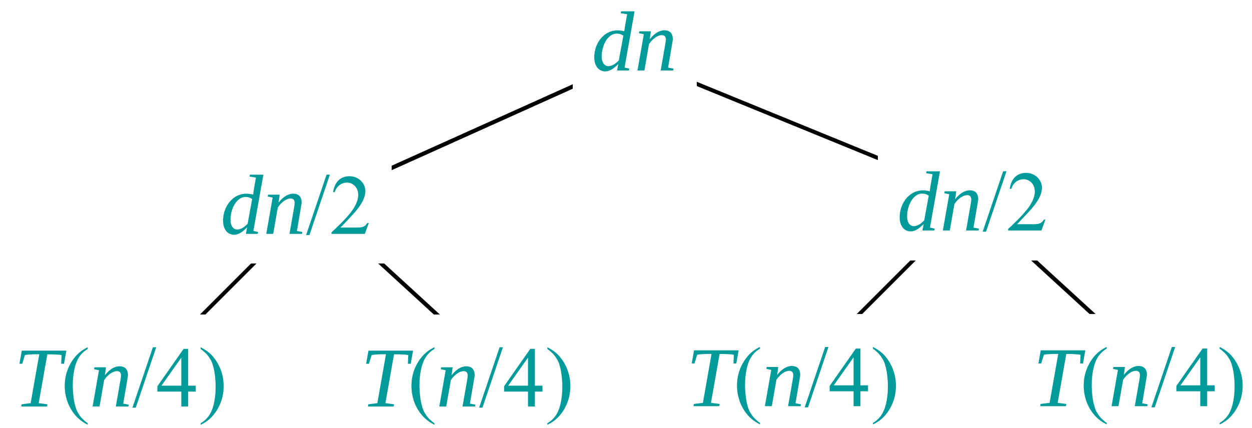
Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

$$dn$$

$$T(n/2) \qquad\qquad T(n/2)$$

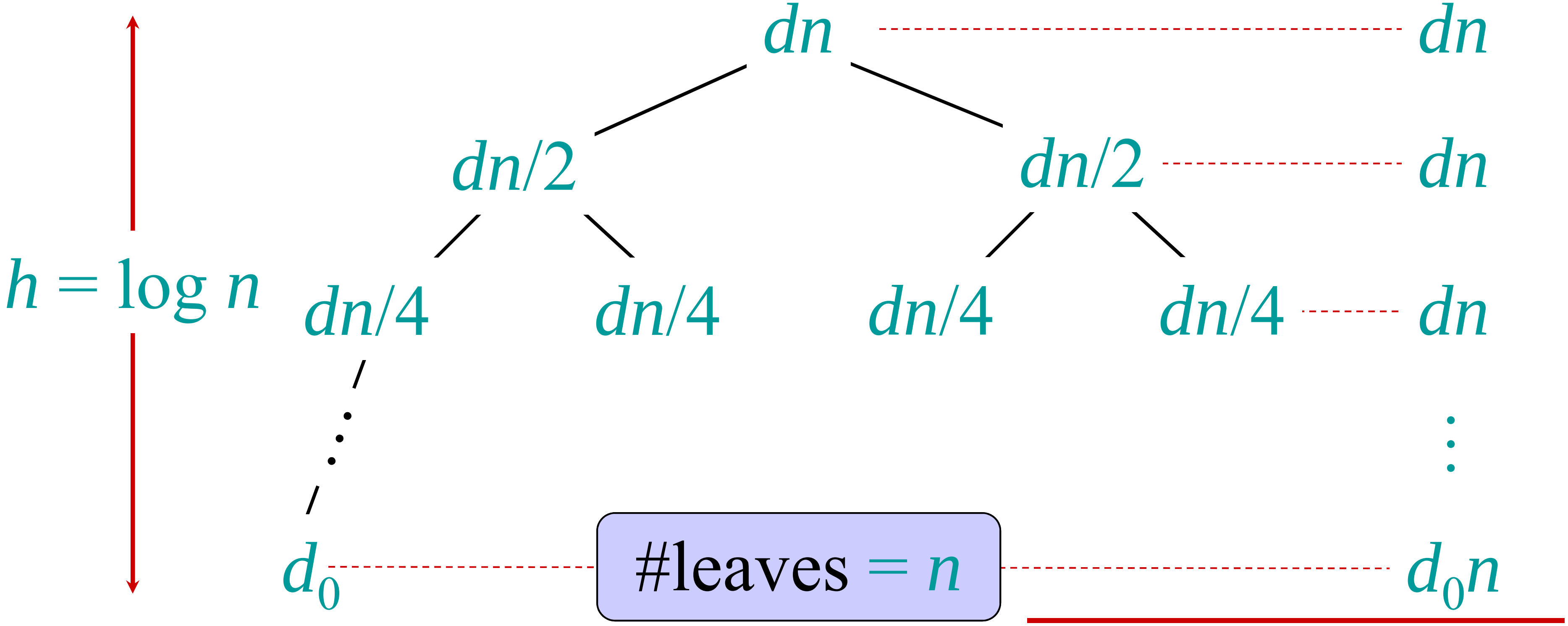
Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.



Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.



Total $dn \log n + d_0 n$

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- It is good for generating **guesses** of what the runtime could be.

But: Need to **verify** that the guess is correct.
$\rightarrow$ Induction (substitution method)

Substitution method

The most general method to solve a recurrence (prove O and Ω separately):

1. ***Guess*** the form of the solution:
   (e.g. using recursion trees, or expansion)
2. ***Verify*** by induction (inductive step).
3. ***Solve*** for O-constants n_0 and c (base case of induction)

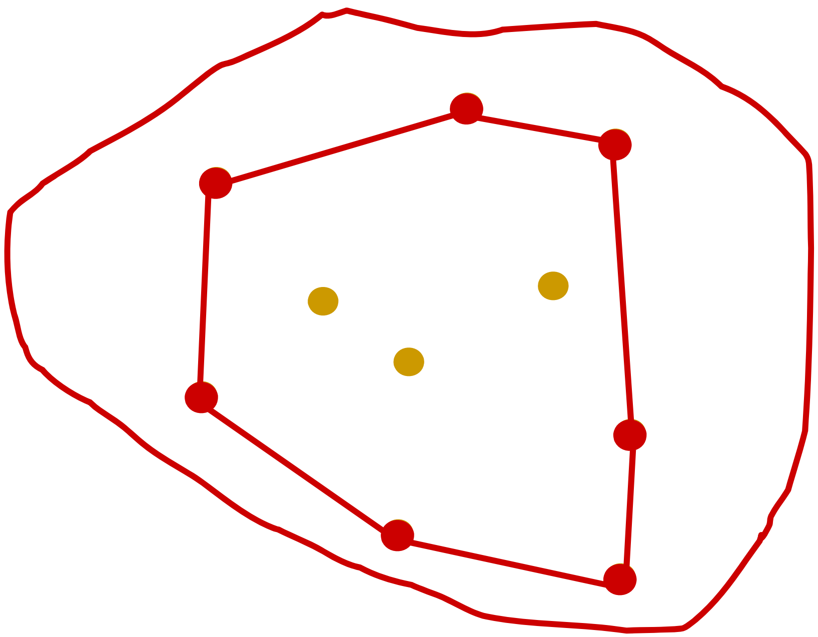
Convex Hull Problem

- Given a set of pins on a pinboard

  and a rubber band around them.

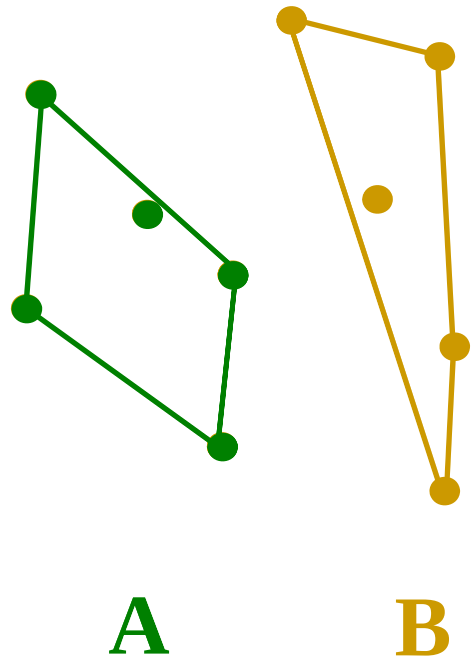
  How does the rubber band look when it snaps tight?

- The convex hull of a point set is one of the simplest shape approximations for a set of points.



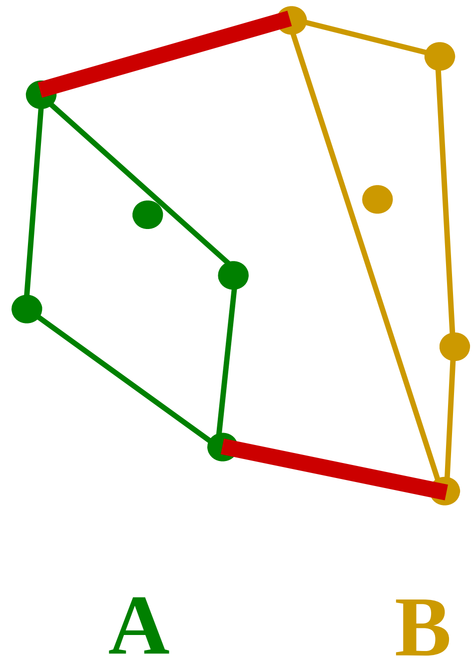
Convex Hull: Divide & Conquer

- Preprocessing: sort the points by x-coordinate
- Divide the set of points into two sets **A** and **B**:
  - **A** contains the left $\lfloor n/2 \rfloor$ points,
  - **B** contains the right $\lceil n/2 \rceil$ points
- Recursively compute the convex hull of **A**
- Recursively compute the convex hull of **B**
- Merge the two convex hulls



Merging

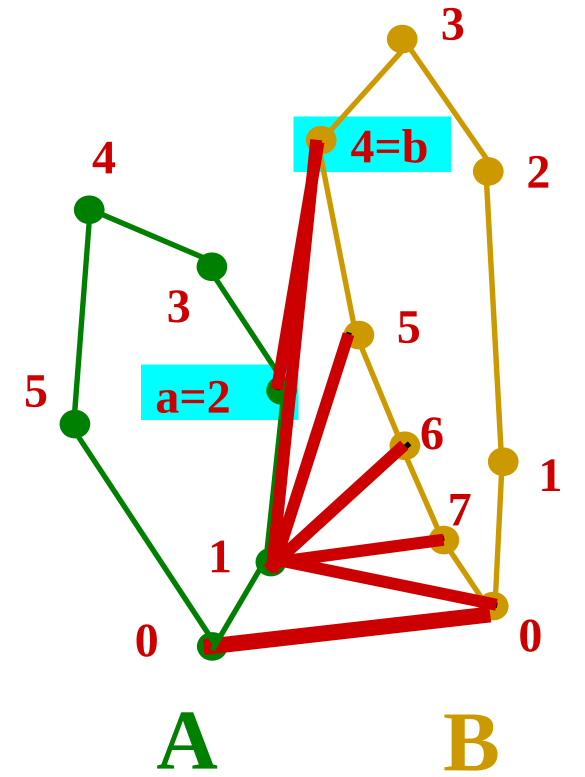
- **Find upper and lower tangent**
- With those tangents the convex hull of A∪B can be computed from the convex hulls of A and the convex hull of B in O(n) linear time

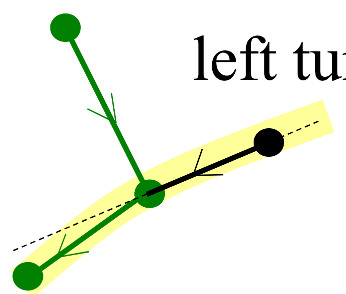


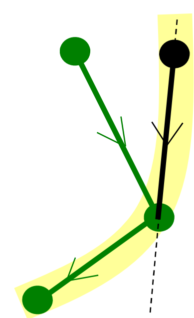
Finding the lower tangent

```
a = rightmost point of A
b = leftmost point of B
while T=ab not lower tangent to both
        convex hulls of A and B do{
      while T not lower tangent to
       convex hull of A do{
          a=a-1
      }
      while T not lower tangent to
       convex hull of B do{
          b=b+1
       }
}
```

check with orientation test




left turn


right turn

Convex Hull: Runtime

• Preprocessing: sort the points by x-coordinate	$O(n \log n)$ just once
• Divide the set of points into two sets **A** and **B**:	$O(1)$
• **A** contains the left $\lfloor n/2 \rfloor$ points,	
• **B** contains the right $\lceil n/2 \rceil$ points	
• Recursively compute the convex hull of **A**	$T(n/2)$
• Recursively compute the convex hull of **B**	$T(n/2)$
• Merge the two convex hulls	$O(n)$

Convex Hull: Runtime

- Runtime Recurrence:

  $T(n) = 2\,T(n/2) + dn$

- Solves to $T(n) = \Theta(n \log n)$

Powering a number

Problem: Compute a^n, where $n \in \boldsymbol{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm: **(recursive squaring)**

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1) \;\Rightarrow\; T(n) = \Theta(\log n)\ .$$