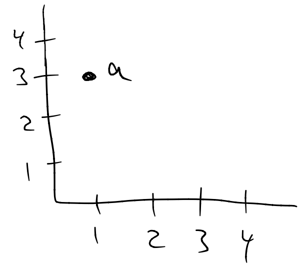
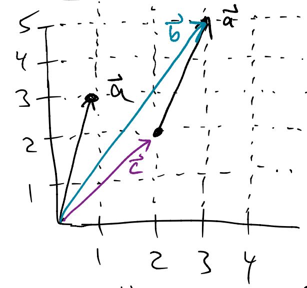
Some geometry and linear algebra

Points in the plane:



$a = (1,3)$

Vectors in the plane:



A vector (arrow) has a length and a direction.

$\Rightarrow$ the notion of a vector is a generalization of a point.

$\vec{a} = \begin{pmatrix}1\\3\end{pmatrix}$

$\vec{b} = \begin{pmatrix}3\\5\end{pmatrix}$

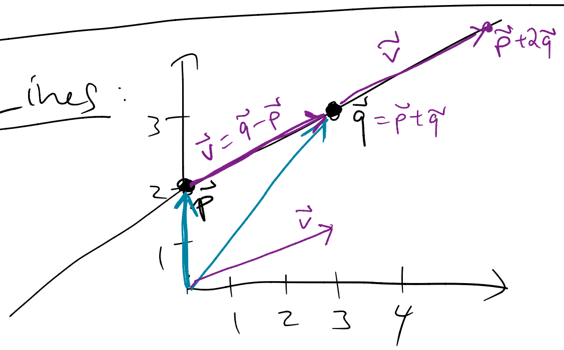
$\vec{c} = \begin{pmatrix}2\\2\end{pmatrix}$

We have: $\vec{b} = \vec{c} + \vec{a}$

$\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}2+1\\2+3\end{pmatrix}$

Or in other words: $\vec{a} = \vec{b} - \vec{c}$

Lines:



$y = mx + n$

$y = \frac{1}{3}x + 2$

line $L = \left\{ \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = \frac{1}{3}x + 2 \right\}$

But: How to represent vertical lines?

$\vec{p} = \begin{pmatrix}0\\2\end{pmatrix}$

$\vec{q} = \begin{pmatrix}3\\3\end{pmatrix}$ $\quad \vec{v} = \vec{q} - \vec{p} = \begin{pmatrix}3\\3\end{pmatrix} - \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$

Another way to describe this line:

$$L = \{ \underbrace{(1-s)\vec{p} + s\cdot\vec{q}}_{\text{affine combination}} \mid s \in \mathbb{R} \} = \{ \vec{p} + s\cdot \underbrace{(\vec{q}-\vec{p})}_{=:\vec{v}\ \text{direction vector}} \mid s \in \mathbb{R} \}$$

$$= \{ s_1 \cdot \vec{p} + s_2 \cdot \vec{q} \mid s_1, s_2 \in \mathbb{R} \text{ and } s_1 + s_2 = 1 \}$$

Line Segment $\quad \overline{pq} = \{ (1-s)\vec{p} + s\cdot\vec{q} \mid s \in [0,1] \}$

$s=0$: $\vec{p} + 0\cdot\vec{q} = \vec{p}$

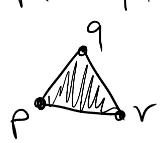
$s=0.4$: ...

$s=1$: $0\cdot\vec{p} + s\cdot\vec{q} = \vec{q}$

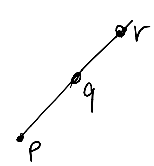
"Linear interpolation between $\vec{p}$ and $\vec{q}$"

Three points $\vec{p}, \vec{q}, \vec{r}$: Triangle

$$\{ s_1\vec{p} + s_2\cdot\vec{q} + s_3\vec{r} \mid s_1, s_2, s_3 \in [0,1] \text{ and } s_1 + s_2 + s_3 = 1 \}$$



unless

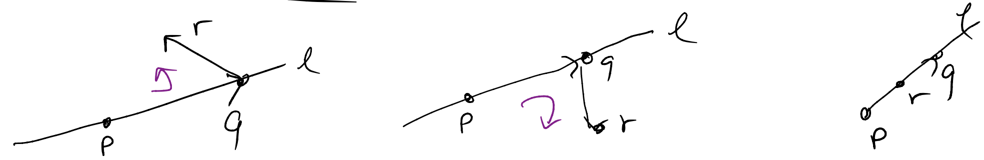


p, q, r lie on a line ("are linearly dependent")

n points $p_1, \dots, p_n$: $CH(\{p_1, \dots, p_n\}) = \left\{ \sum_{i=1}^{n} s_i \vec{p_i} \;\middle|\; s_i \in [0,1] \text{ and } \sum_{i=1}^{n} s_i = 1 \right\}$

Alternative characterization. Need to prove equivalence to our other definition.

Orientation test / halfplane test:



Left turn (positive)
r lies on left side of l

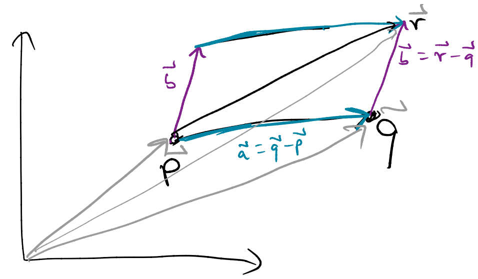
Right turn (negative)
r lies on right side of l

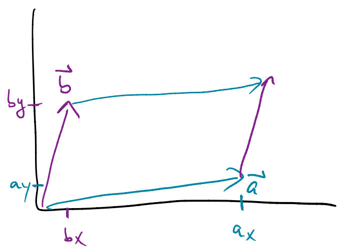
Zero

$\boxed{\text{Use the sign of the oriented area of } \Delta pqr.}$

How to compute this signed area?

- $\frac{1}{2}$ (signed) area of parallelogram





sign of area = { positive, if rotation from a to b along smaller angle is counter clockwise
0, if a and b lie on the same line
negative, otherwise

$$\text{signed area of parallelogram} = \det\begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix} = a_x \cdot b_y - a_y \cdot b_x$$

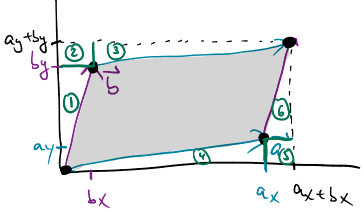
$$\Rightarrow \text{signed area of } \Delta pqr = \frac{1}{2}\det\begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix} = \frac{1}{2}\det\begin{pmatrix} q_x - p_x & r_x - q_x \\ q_y - p_y & r_y - q_y \end{pmatrix}$$

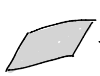
$$= \frac{1}{2}\cdot\left((q_x - p_x)\cdot(r_y - q_y) - (q_y - p_y)\cdot(r_x - q_x)\right)$$

$$= \frac{1}{2}\left(q_x r_y - q_y r_x - (p_x r_y - p_y r_x) + p_x q_y - q_x p_y\right)$$

$$= \frac{1}{2}\det\begin{pmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ p_x & q_x & r_x \\ p_y & q_y & r_y \end{pmatrix} = \frac{1}{2}\cdot\left(\begin{pmatrix} 1 \\ p_x \\ p_y \end{pmatrix} \times \begin{pmatrix} 1 \\ q_x \\ q_y \end{pmatrix}\right)\circ\begin{pmatrix} 1 \\ r_x \\ r_y \end{pmatrix}$$

$$= \frac{1}{2}\cdot\begin{pmatrix} p_x q_y - p_y q_x \\ -(1\cdot q_y - 1\cdot p_y) \\ 1\cdot q_x - 1\cdot p_x \end{pmatrix}\circ\begin{pmatrix} 1 \\ r_x \\ r_y \end{pmatrix}$$



Area of  $= \underset{(1)}{(a_x+b_x)(a_y+b_y)} - \frac{1}{2}b_x b_y - \underset{(2)}{b_x a_y}$

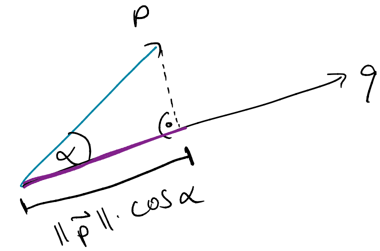
$- \underset{(3)}{\frac{1}{2}a_x a_y} - \underset{(4)}{\frac{1}{2}a_x a_y} - \underset{(5)}{b_x a_y} - \underset{(6)}{\frac{1}{2}b_y b_x}$

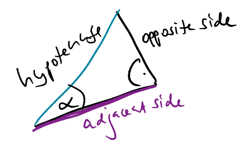
$= a_x b_y - b_x a_y$

Scalar product: $\vec{p}\cdot\vec{q} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = p_1 q_1 + p_2 q_2$

Length of a vector: $\|\vec{p}\| = \sqrt{\vec{p}\cdot\vec{p}}$

Geometric interpretation $\vec{p}\cdot\vec{q} = \|\vec{q}\| \cdot \underbrace{\|\vec{p}\| \cdot \cos\alpha}_{\text{projection of } \vec{p} \text{ onto } \vec{q}}$





$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\Rightarrow \vec{p}\cdot\vec{q} = \|\vec{q}\| \cdot \|\vec{p}\| \cdot \cos\alpha$

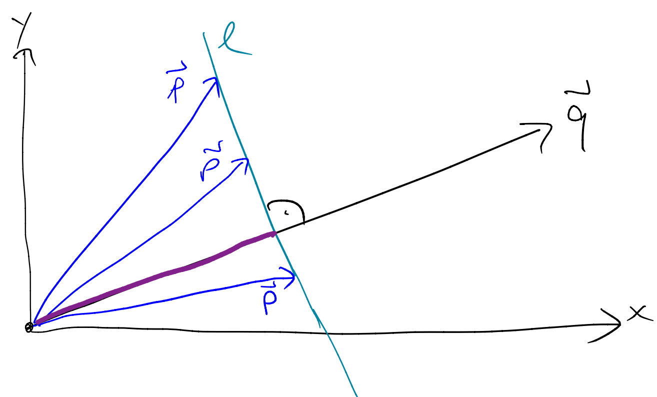
$\Rightarrow \boxed{\alpha = \cos^{-1} \frac{\vec{p}\cdot\vec{q}}{\|\vec{q}\| \cdot \|\vec{p}\|}}$ Works also in $\mathbb{R}^d$

Another characterization of a line ℓ:

$\ell = \ell_{\vec{q}, q} = \{ \vec{p} \in \mathbb{R}^2 \mid \vec{p}\cdot\vec{q} = b \} = \{ \vec{p} \in \mathbb{R}^2 \mid \underbrace{\|\vec{p}\| \cdot \cos\alpha}_{\text{projection of } \vec{p} \text{ onto } \vec{q}} = \underbrace{\frac{b}{\|\vec{q}\|}}_{\text{constant}} \}$

fix

Set of all vectors whose projection onto $\vec{q}$ has fixed length



$\Rightarrow$ Same definition works in $\mathbb{R}^d$ to characterize hyperplanes using their <u>normal vector</u> $\vec{q}$.