CMPS 3130/6130 Computational Geometry
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Delaunay Triangulations I
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Based on:
Computational Geometry: Algorithms and Applications
Triangulation

• Let $P = \{p_1, ..., p_n\} \subseteq \mathbb{R}^2$ be a finite set of points in the plane.
• A **triangulation of** $P$ is a simple, plane (i.e., planar embedded), connected graph $T=(P,E)$ such that
  – every edge in $E$ is a line segment,
  – the outer face is bounded by edges of $\text{CH}(P)$,
  – all inner faces are triangles.
**Dual Graph**

- Let $G = (V, E)$ be a plane graph. The dual graph $G^*$ has
  - a vertex for every face of $G$,
  - an edge for every edge of $G$, between the two faces incident to the original edge.
Delaunay Triangulation

- Let $G$ be the plane graph for the Voronoi diagram $VD(P)$. Then the dual graph $G^*$ is called the **Delaunay Triangulation** $DT(P)$.

Can be stored as an abstract graph, without geometric information. (No such obvious storing scheme for $VD(P)$.)

- If $P$ is in general position (no three points on a line, no four points on a circle) then every inner face of $DT(P)$ is indeed a triangle.
Straight-Line Embedding

- **Lemma:** DT(\(P\)) is a plane graph, i.e., the straight-line edges do not intersect.

- **Proof:**
  - \(pp'\) is an edge of DT(\(P\)) ⇔ There is an empty closed disk \(D_p\) with \(p\) and \(p'\) on its boundary, and its center \(c\) on the bisector.
  - Let \(qq'\) be another Delaunay edge that intersects \(pp'\)
    \(⇒ q\) and \(q'\) lie outside of \(D_p\), therefore \(qq'\) also intersects \(pc\) or \(p'c\).
  - Similarly, \(pp'\) also intersects \(qc'\) or \(q'c'\)

\(⇒ (pc\) or \(p'c')\) and \((qc'\) or \(q'c')\) intersect
\(⇒\) The edges are not in different Voronoi cells
\(⇒\) Contradiction
Characterization I of DT(P)

- **Lemma**: Let $p, q, r \in P$ let $\Delta$ be the triangle they define. Then the following statements are equivalent:
  a) $\Delta$ belongs to $DT(P)$
  b) The circumcenter of $\Delta$ is a vertex in $VD(P)$
  c) The circumcircle of $\Delta$ is empty (i.e., contains no other point of $P$)

- **Characterization I**: Let $T$ be a triangulation of $P$. Then $T = DT(P) \iff$ The circumcircle of any triangle in $T$ is empty.
Illegal Edges

- **Definition:** Let \( p_i, p_j, p_k, p_l \in P \). Then \( p_i p_j \) is an illegal edge \( \iff \) \( p_l \) lies in the interior of the circle through \( p_i, p_j, p_k \).

- **Lemma:** Let \( p_i, p_j, p_k, p_l \in P \). Then \( p_i p_j \) is illegal \( \iff \min_{1 \leq i \leq 6} \alpha_i < \min_{1 \leq i \leq 6} \alpha'_i \).

- **Theorem (Thales):** Let \( a, b, p, q \) be four points on a circle, and let \( r \) be inside and let \( s \) be outside of the circle, such that \( p, q, r, s \) lie on the same side of the line through \( a, b \). Then \( \angle a, s, b < \angle a, q, b = \angle a, p, b < \angle a, r, b \).
Characterization II of DT(P)

- **Definition:** A triangulation is called legal if it does not contain any illegal edges.

- **Characterization II:** Let $T$ be a triangulation of $P$. Then $T = DT(P) \iff T$ is legal.

- **Algorithm Legal_Triangulation($T$):**
  
  **Input:** A triangulation $T$ of a point set $P$
  
  **Output:** A legal triangulation of $P$

  while $T$ contains an illegal edge $\overline{p_ip_j}$ do
  
  //Flip $\overline{p_ip_j}$
  
  Let $p_i, p_j, p_k, p_l$ be the quadrilateral containing $\overline{p_ip_j}$
  
  Remove $\overline{p_ip_j}$ and add $\overline{p_kp_l}$

  return $T$

  **Runtime analysis:**
  
  - In every iteration of the loop the angle vector of $T$ (all angles in $T$ sorted by increasing value) increases
  
  - With this one can show that a flipped edge never appears again
  
  - There are $O(n^2)$ edges, therefore the runtime is $O(n^2)$
Characterization III of DT(P)

• **Definition:** Let $T$ be a triangulation of $P$ and let $\alpha_1$, $\alpha_2$, ..., $\alpha_m$ be the angles of the $m$ triangles in $T$ sorted by increasing value. Then $A(T)=(\alpha_1, \alpha_2, ..., \alpha_m)$ is called the angle vector of $T$.

• **Definition:** A triangulation $T$ is called **angle optimal** $\iff A(T) > A(T')$ for any other triangulation of the same point set $P$.

• Let $T'$ be a triangulation that contains an illegal edge, and let $T''$ be the resulting triangulation after flipping this edge. Then $A(T'') > A(T')$.

• $T$ is angle optimal $\Rightarrow$ $T$ is legal $\Rightarrow T=DT(P)$

• **Characterization III:** Let $T$ be a triangulation of $P$. Then $T=DT(P) \iff T$ is angle optimal.

(If $P$ is not in general position, then any triangulation obtained by triangulating the faces maximizes the minimum angle.)