CMPS 2200 – Fall 2017

Red-black trees

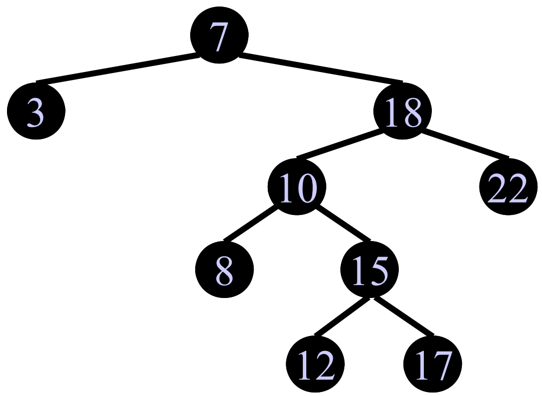
Carola Wenk

Slides courtesy of Charles Leiserson with changes by Carola Wenk

Dynamic Set

A **dynamic set**, or **dictionary**, is a data structure which supports operations

- Insert
- Delete
- Find



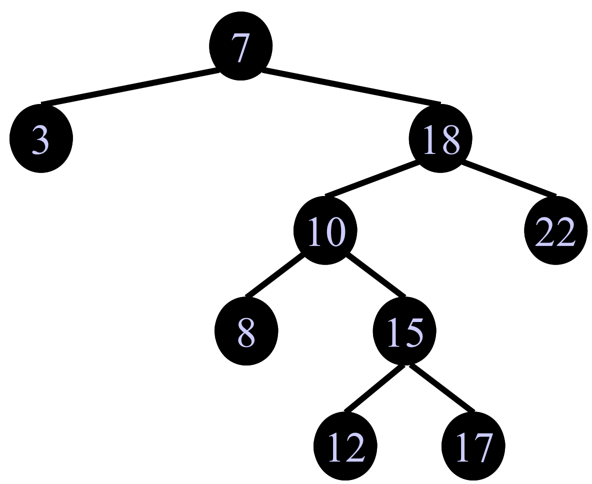
Using **balanced binary search trees** we can implement a dictionary data structure such that each operation takes $O(\log n)$ time.

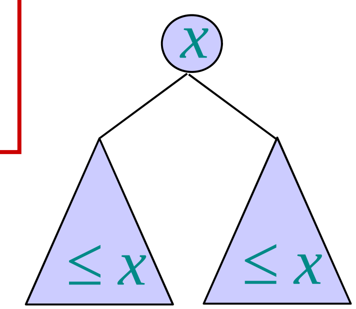
Search Trees

- A binary search tree is a binary tree. Each node stores a key. The tree fulfills the **binary search tree property**:

For every node x holds:

- $y \leq x$, for all y in the subtree left of x
- $x < y$, for all y in the subtree right of x

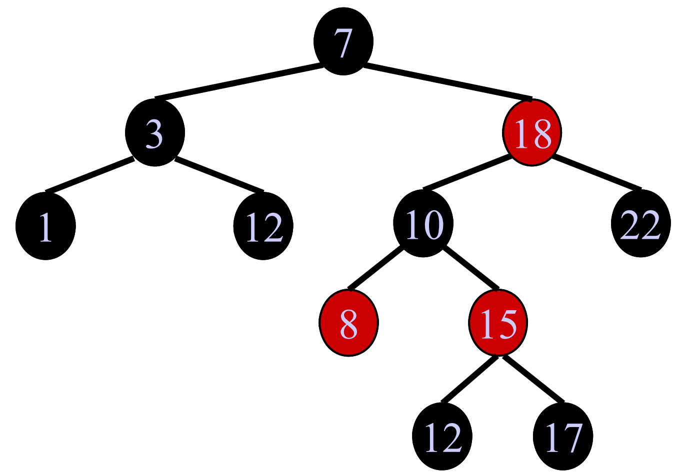




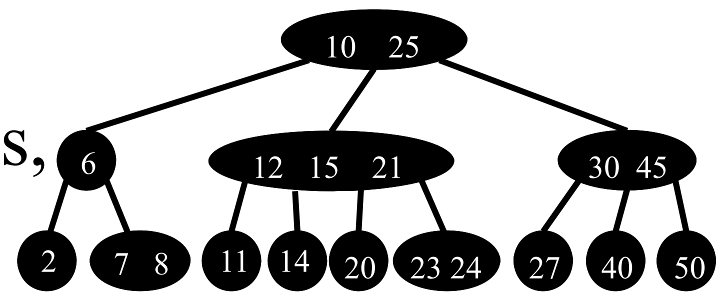
Search Trees

Different variants of search trees:

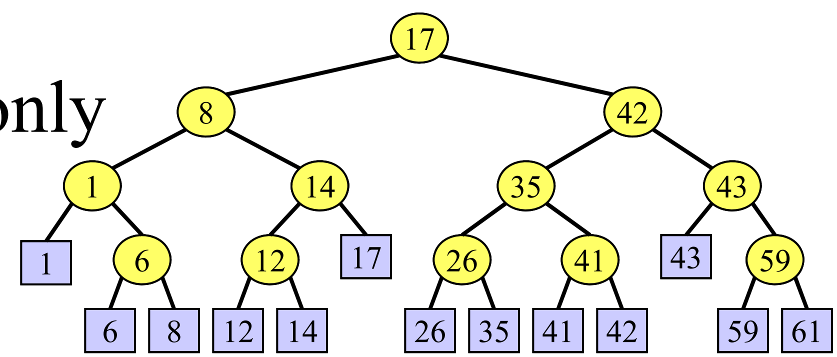
• Balanced search trees (guarantee height of O(log n) for n elements)



• k-ary search trees (such as B-trees, 2-3-4-trees)



• Search trees that store keys only in leaves, and store copies of keys as split-values in internal nodes



Balanced search trees

Balanced search tree: A search-tree data structure for which a height of $O(\log n)$ is guaranteed when implementing a dynamic set of n items.

Examples:

- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees

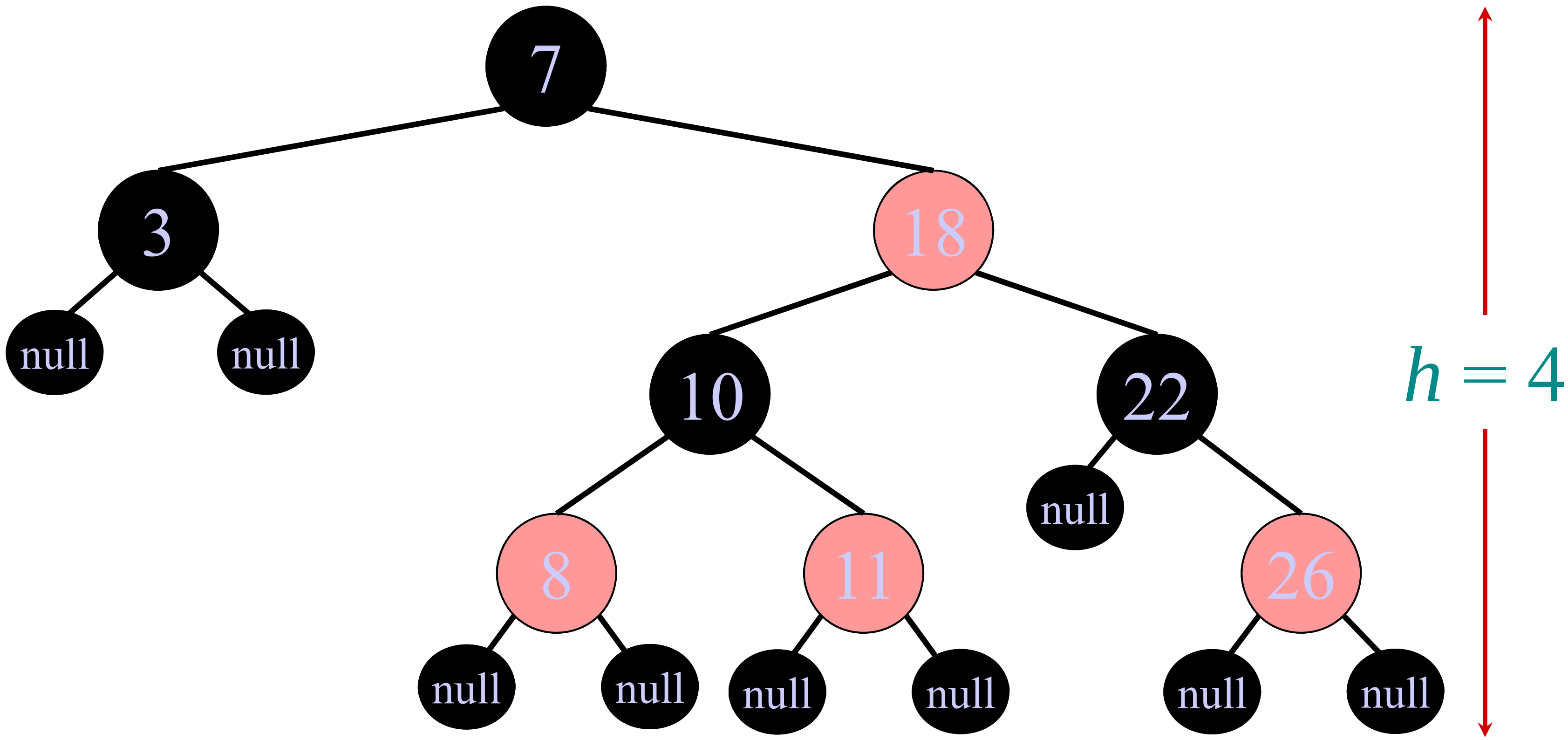
Red-black trees

This data structure requires an extra one-bit color field in each node.

Red-black properties:

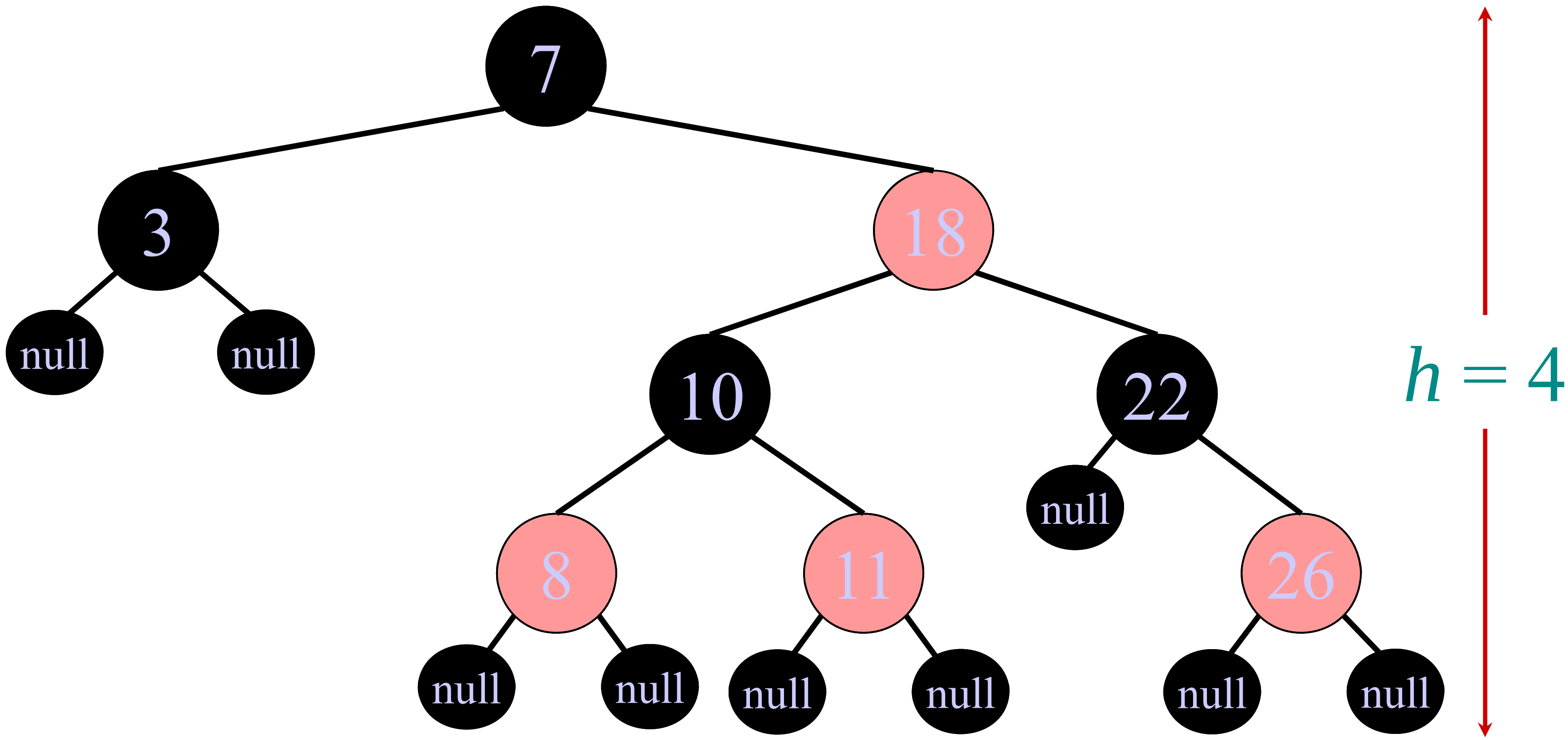
1. Every node is either red or black.
2. The root is black.
3. The leaves (null's) are black.
4. If a node is red, then both its children are black.
5. All simple paths from any node x, excluding x, to a descendant leaf have the same number of black nodes = black-height(x).

Example of a red-black tree



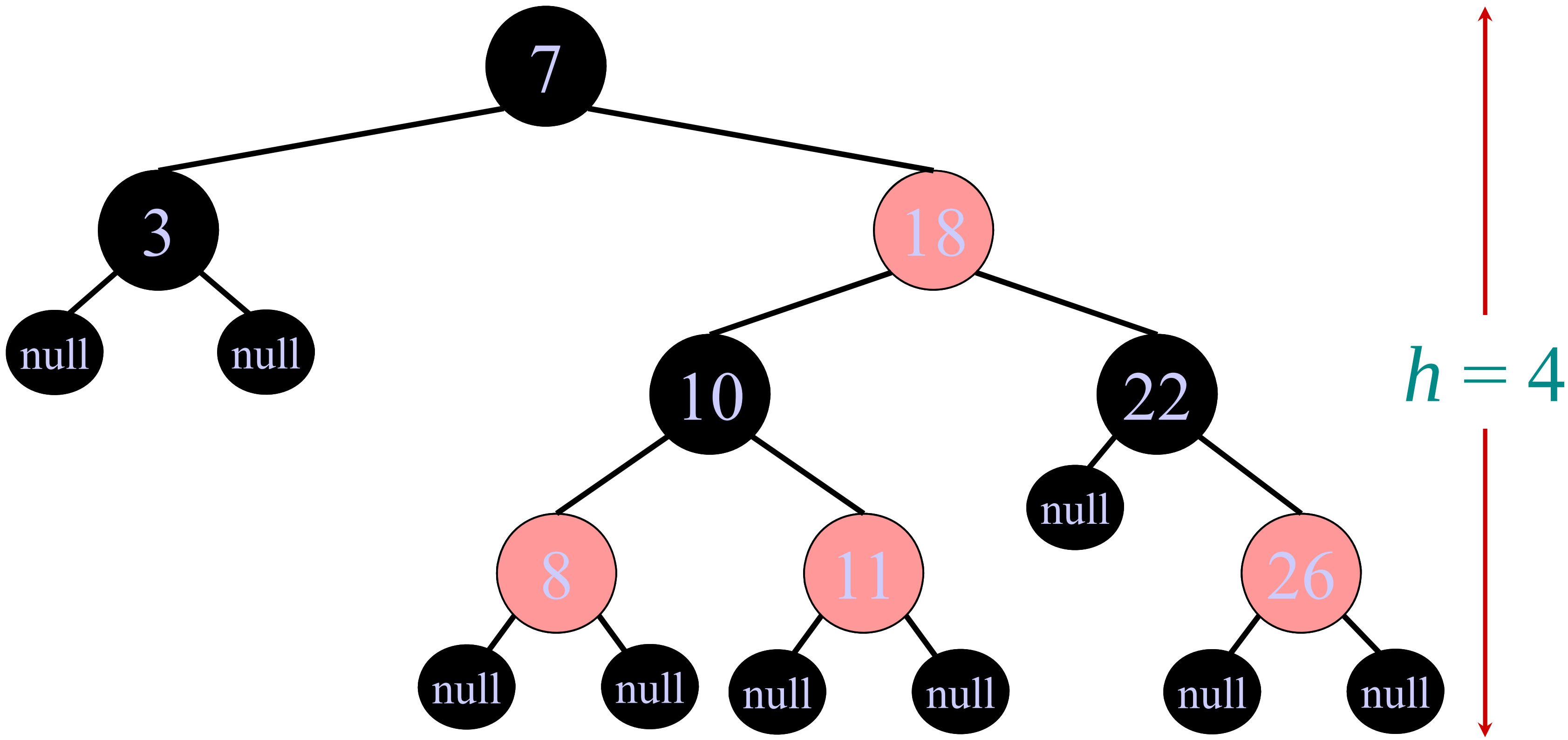
1. Every node is either red or black.

Example of a red-black tree



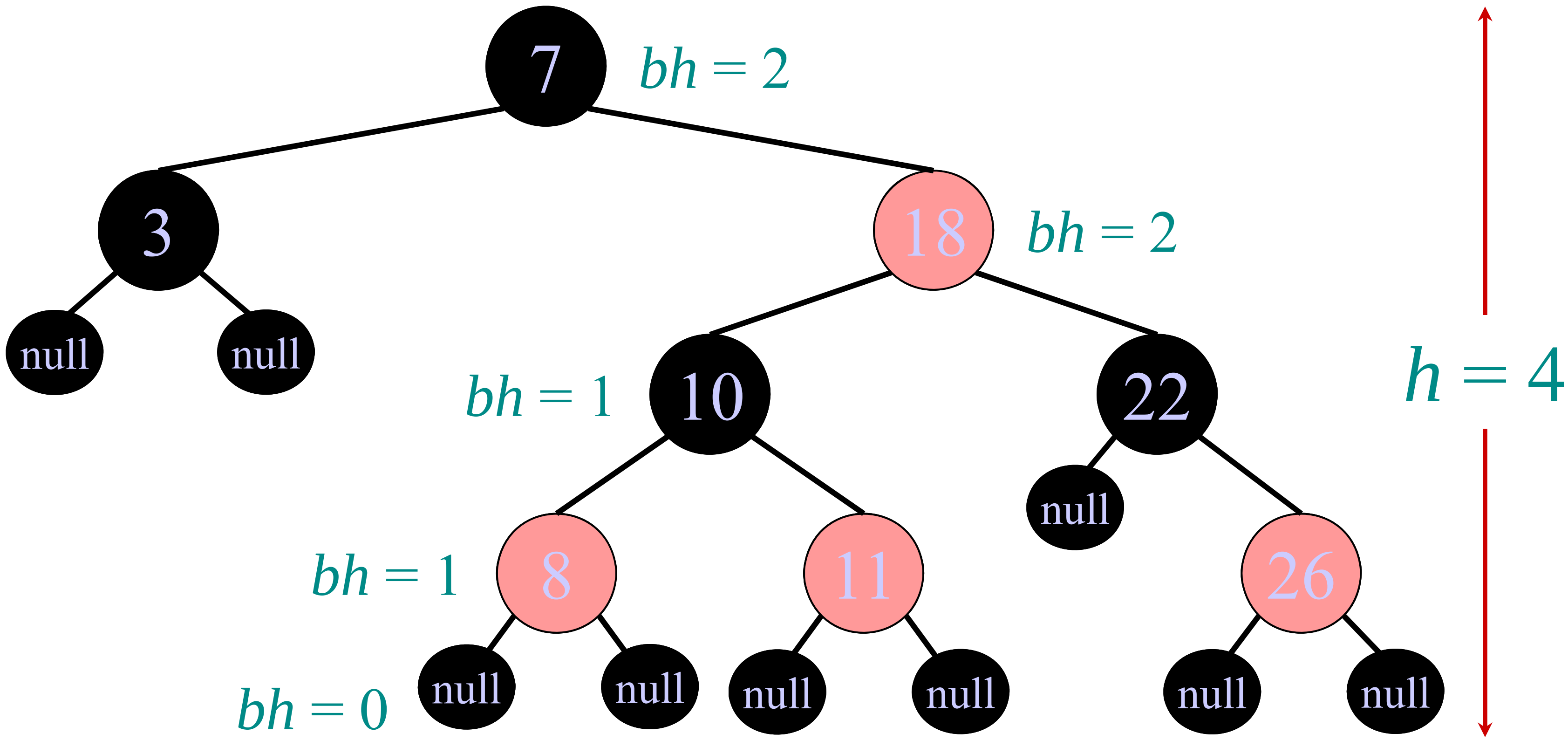
2., 3. The root and leaves (null's) are black.

Example of a red-black tree



4. If a node is red, then both its children are black.

Example of a red-black tree



5. All simple paths from any node x, excluding x, to a descendant leaf have the same number of black nodes = *black-height*(x).

Height of a red-black tree

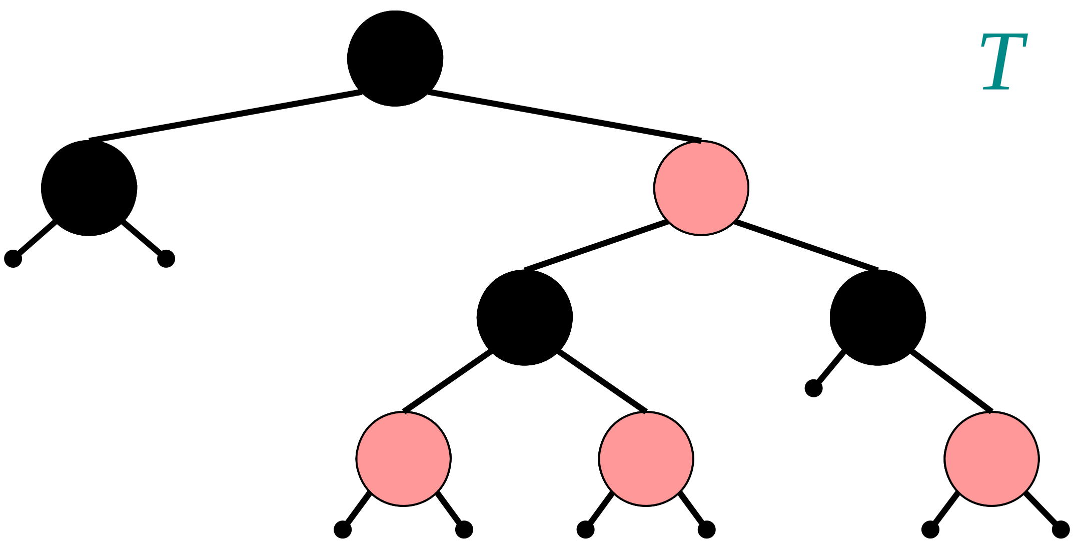
Theorem. A red-black tree with n keys has height

$$h \leq 2\log(n+1).$$

Proof.

INTUITION:

- Merge red nodes into their black parents.



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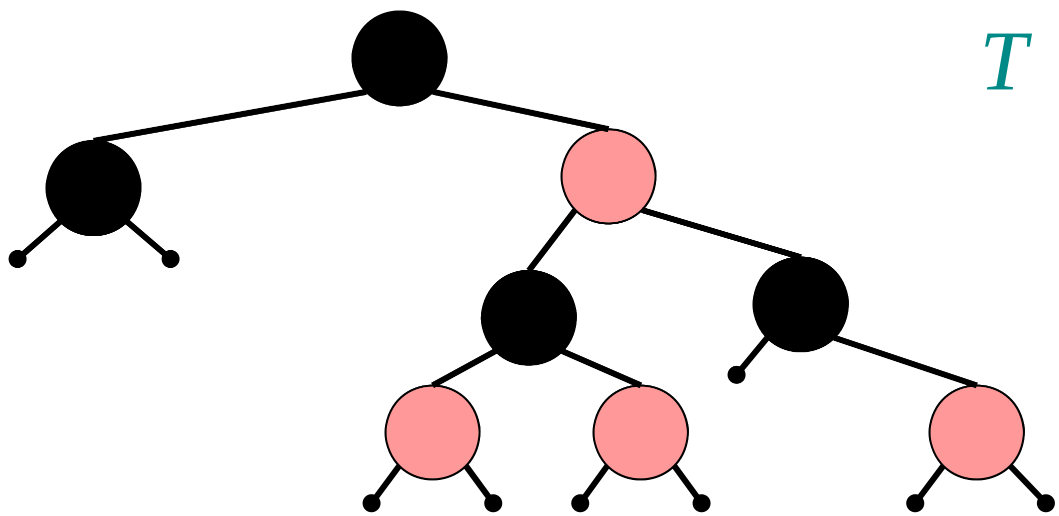

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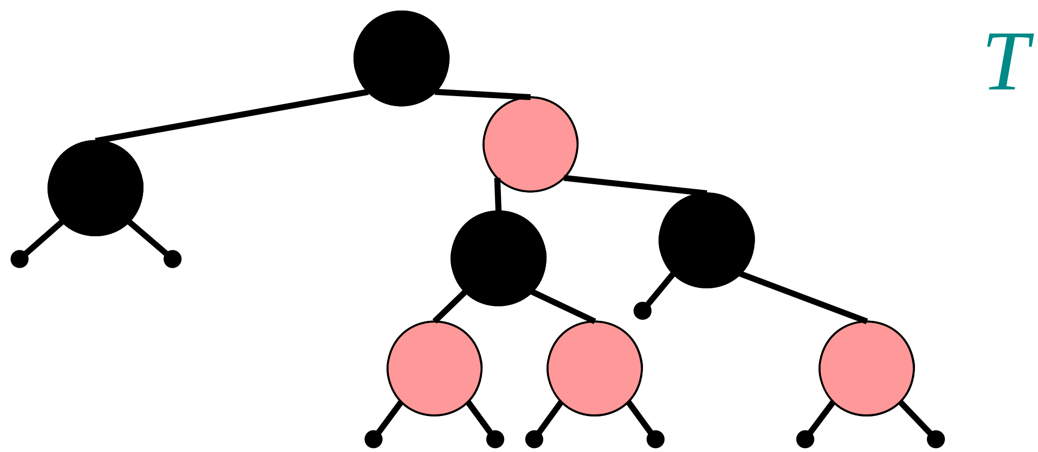

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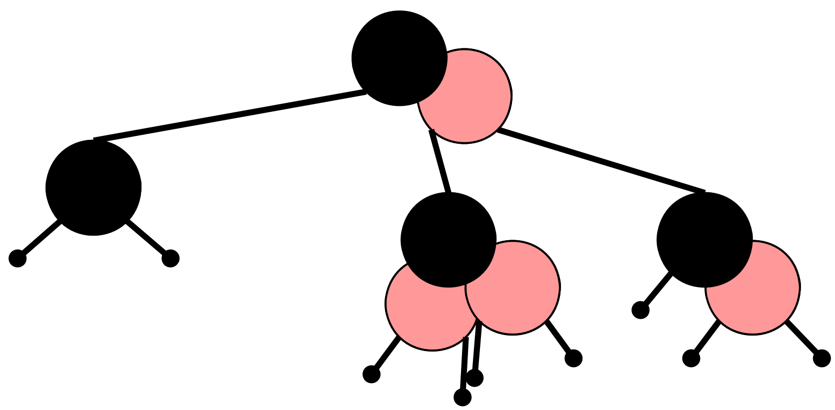


T

Height of a red-black tree

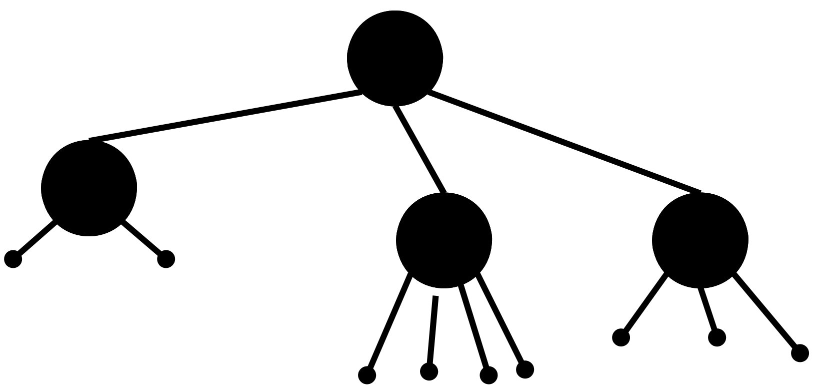
Theorem. A red-black tree with n keys has height

$$h \leq 2 \log(n + 1).$$

Proof.

INTUITION:

- Merge red nodes into their black parents.



T'

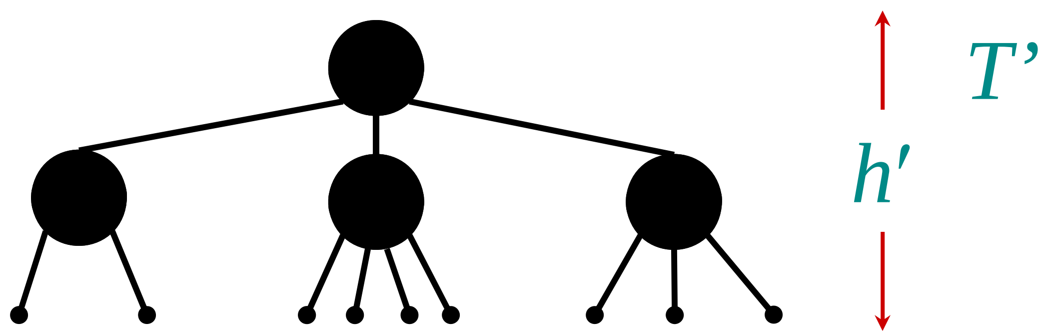
Height of a red-black tree

Theorem. A red-black tree with n keys has height

$$h \le 2 \log(n + 1).$$

Proof.

INTUITION:



- Merge red nodes into their black parents.
- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth h' of leaves.

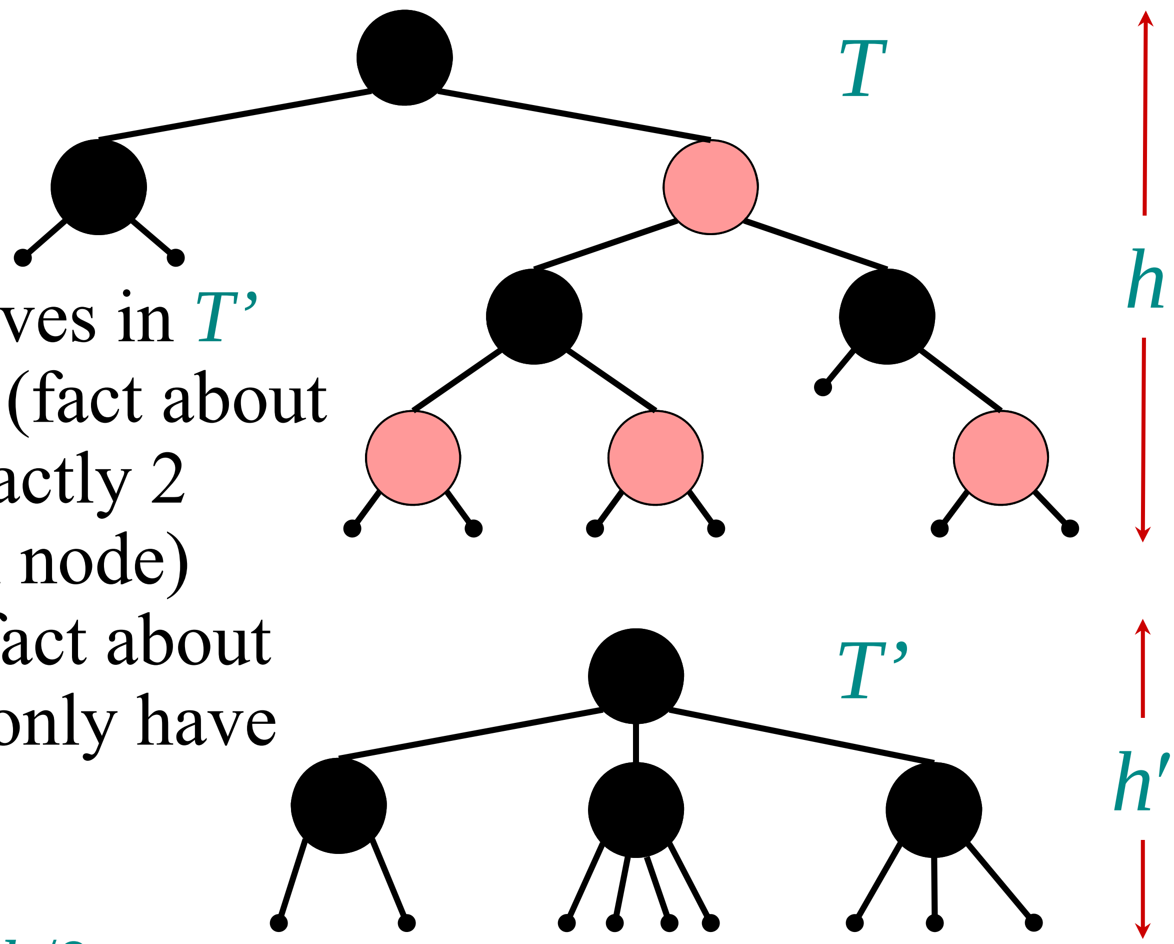
Proof (continued)

- We have $h' \geq h/2$, since at most half the vertices on any path are red.
- # leaves in T = # leaves in T'
- # leaves in $T = n+1$ (fact about binary trees with exactly 2 children per internal node)
- # leaves in $T' \geq 2^{h'}$ (fact about binary trees; T' can only have more)

$\Rightarrow n + 1 \geq 2^{h'}$

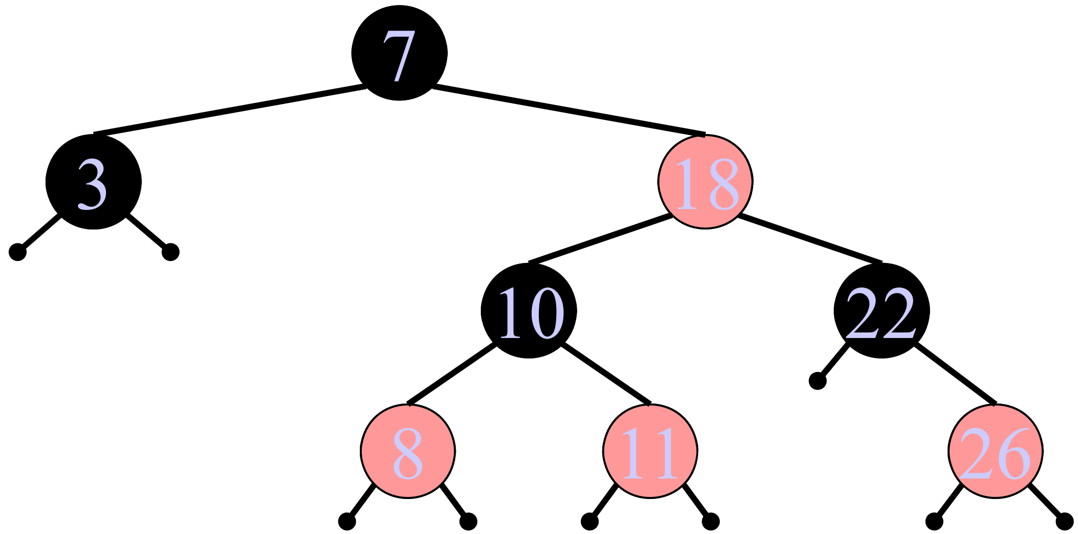
$\Rightarrow \log(n + 1) \geq h' \geq h/2$

$\Rightarrow h \leq 2 \log(n + 1)$.



Query operations

Corollary. The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in $O(\log n)$ time on a red-black tree with n nodes.

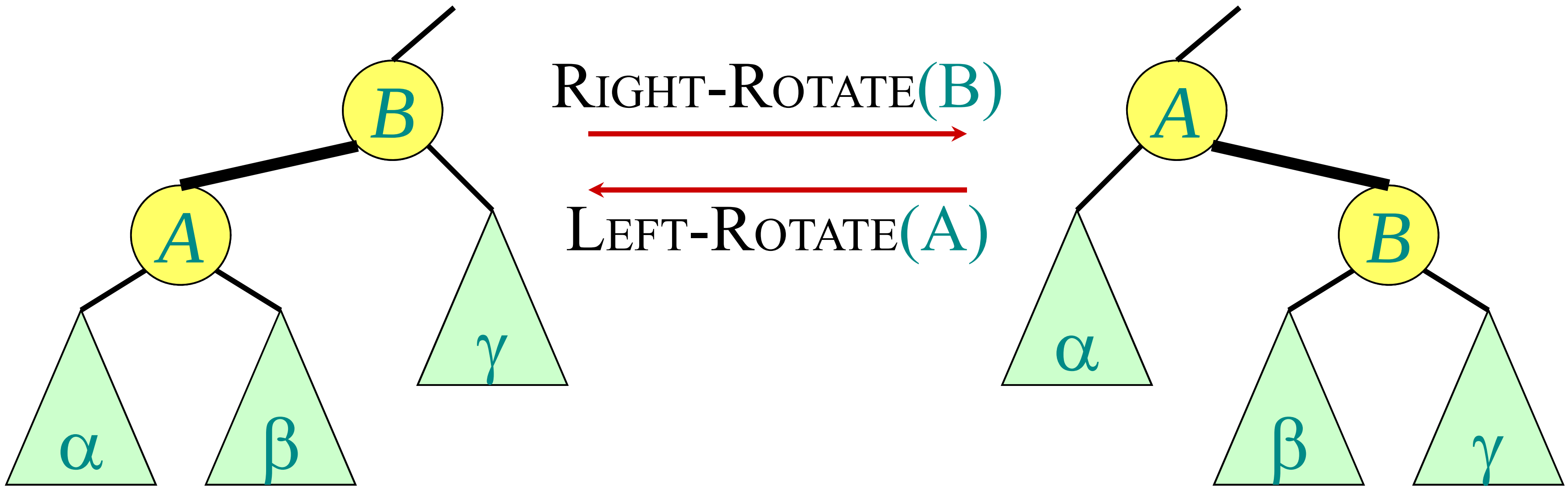


Modifying operations

The operations INSERT and DELETE cause modifications to the red-black tree:

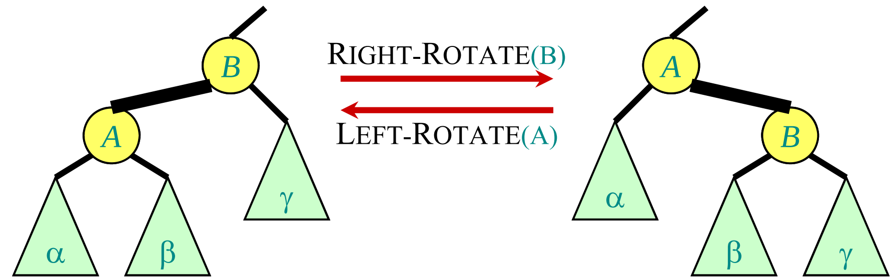
1. the operation itself,
2. color changes,
3. restructuring the links of the tree via ***"rotations"***.

Rotations

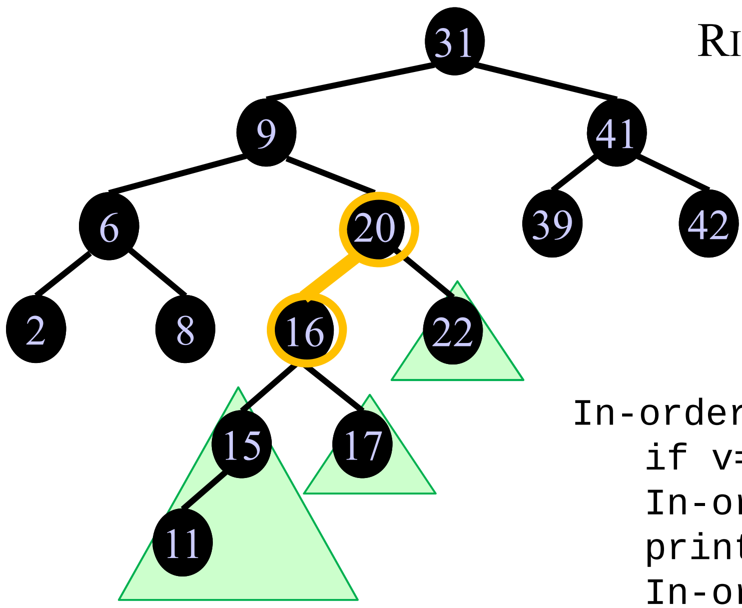


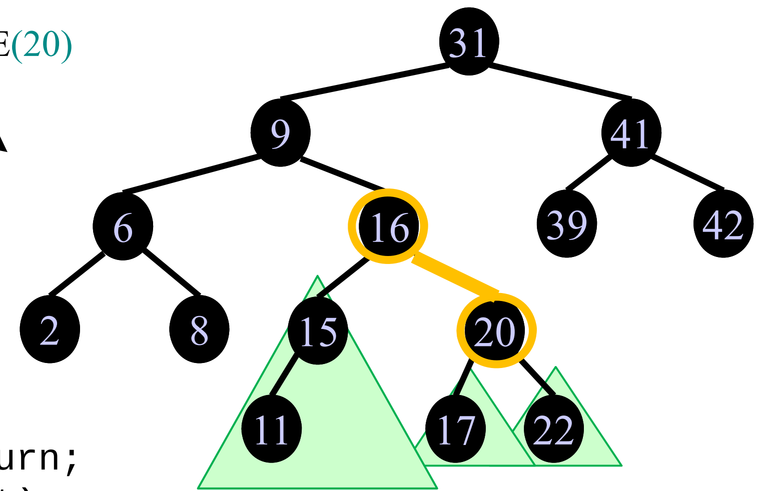
- Rotations maintain the inorder ordering of keys:
  $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \le A \le b \le B \le c$.
- Rotations maintain the binary search tree property
  $\Rightarrow$ Can be applied to any BST, not just red-black trees
- A rotation can be performed in $O(1)$ time.

Rotation Example



RIGHT-ROTATE(20)






```
In-order(v){
    if v==null return;
    In-order(v.left);
    print(v.key+” ”);
    In-order(v.right);
}
```

In-order traversal:
2 6 8 9 11 15 16 17 20 22 31 39 41 42

In-order traversal:
2 6 8 9 11 15 16 17 20 22 31 39 41 42

$\Rightarrow$ Maintains sorted order of keys, and can reduce height

Red-black trees

This data structure requires an extra one-bit color field in each node.

Red-black properties:

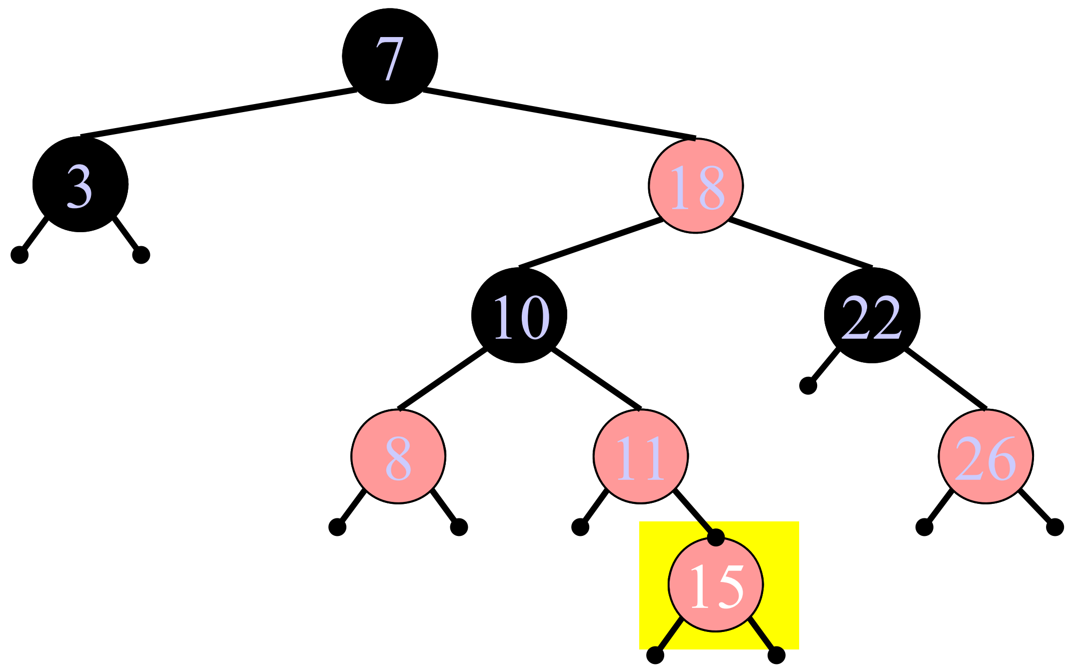
1. Every node is either red or black.
2. The root is black.
3. The leaves (null's) are black.
4. If a node is red, then both its children are black.
5. All simple paths from any node x, excluding x, to a descendant leaf have the same number of black nodes = black-height(x).

Insertion into a red-black tree

IDEA: Insert *x* in tree. Color *x* red. Only red-black property 4 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

Example:

- Insert $x = 15$.



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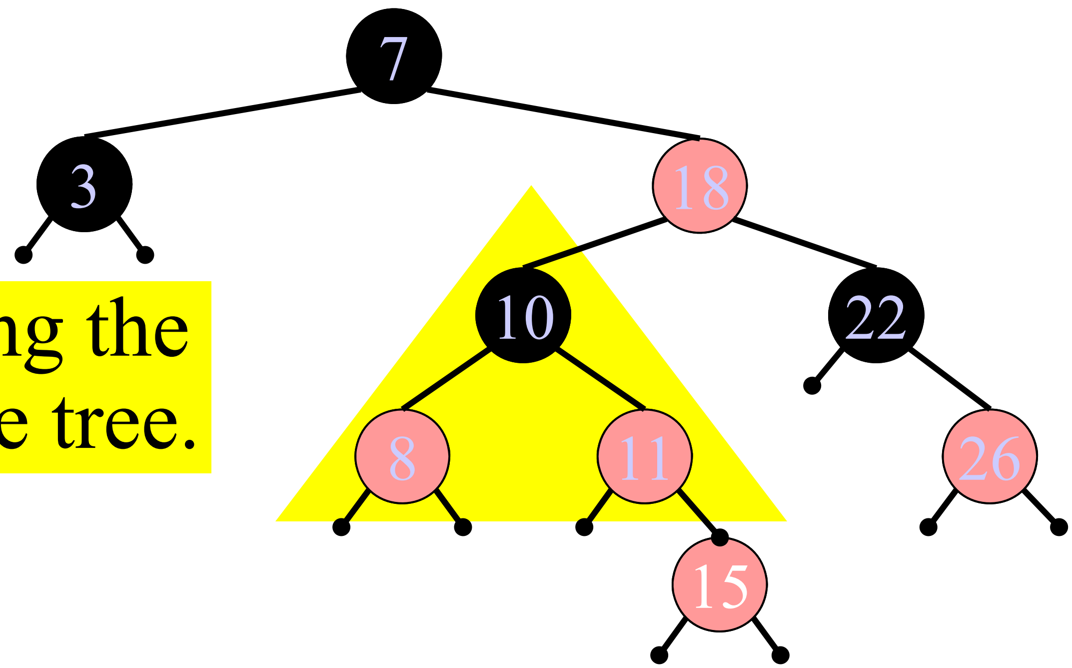
Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.



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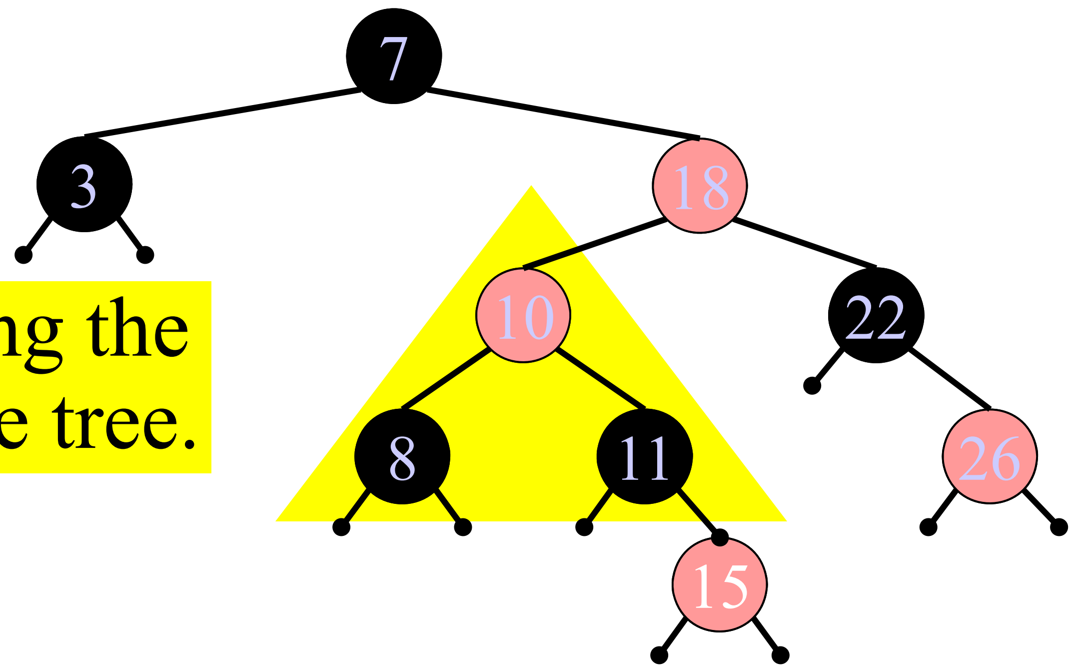

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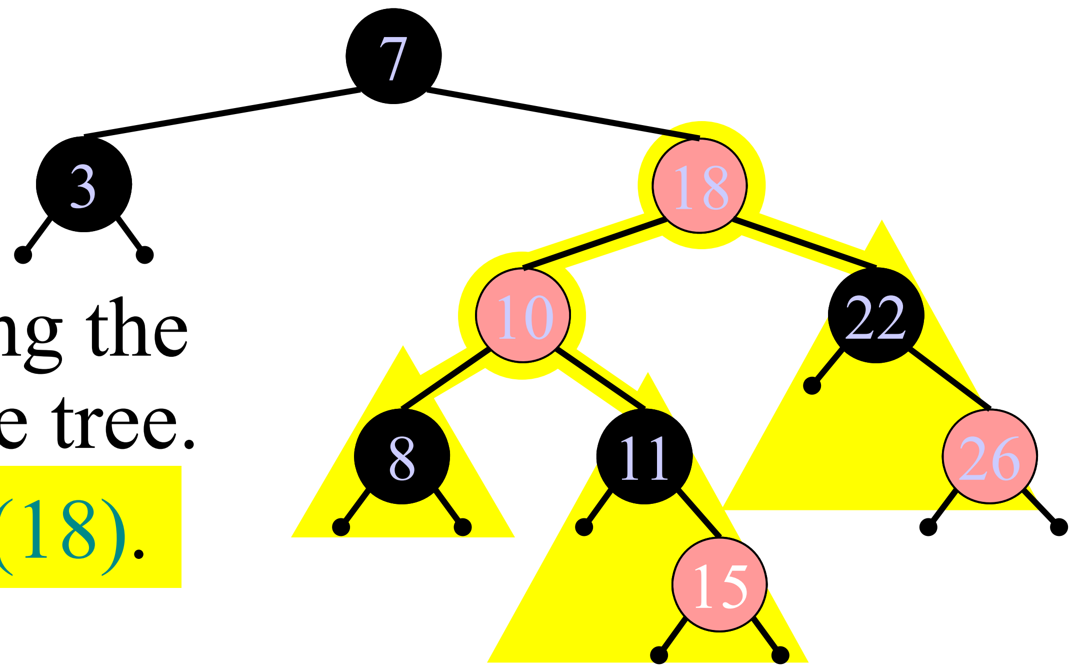
- RIGHT-ROTATE(18).



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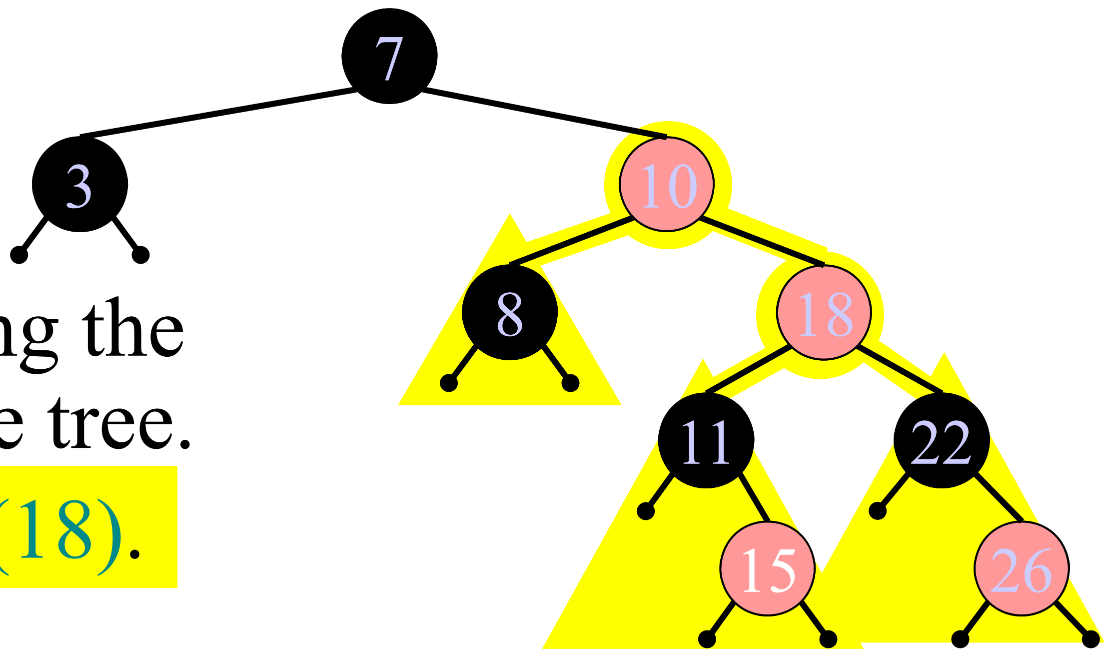

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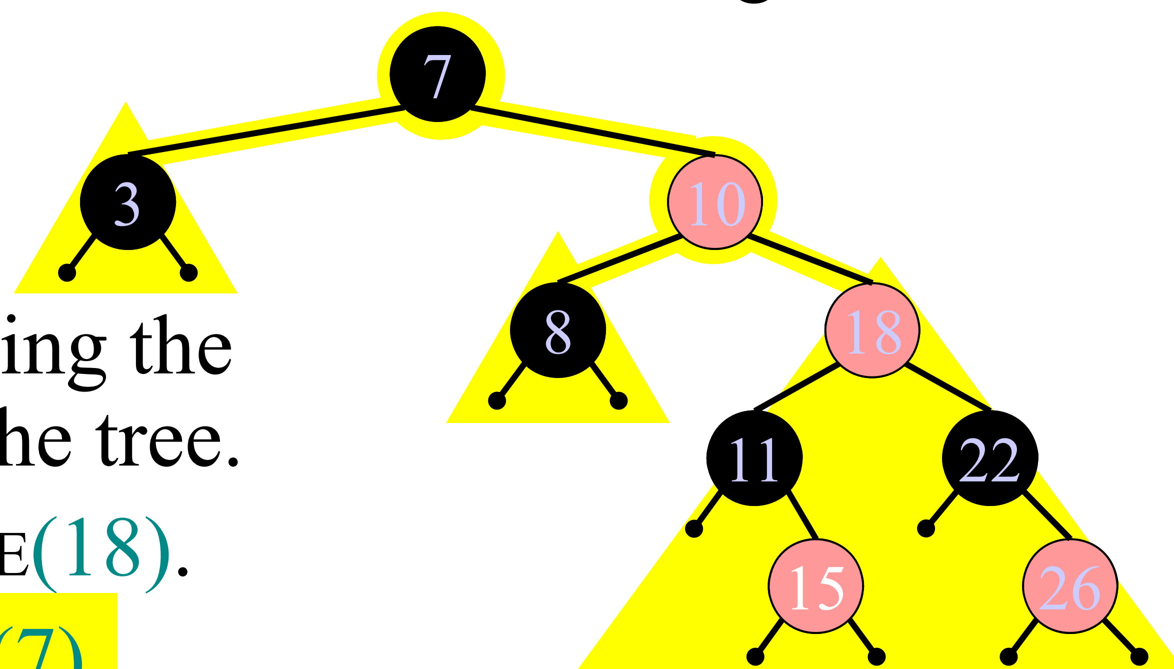
Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7)



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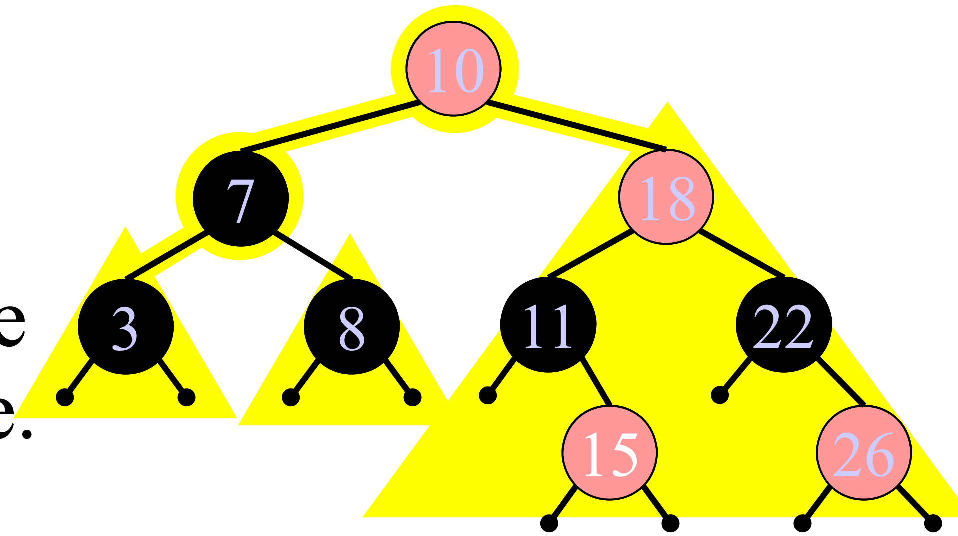
- LEFT-ROTATE(7)



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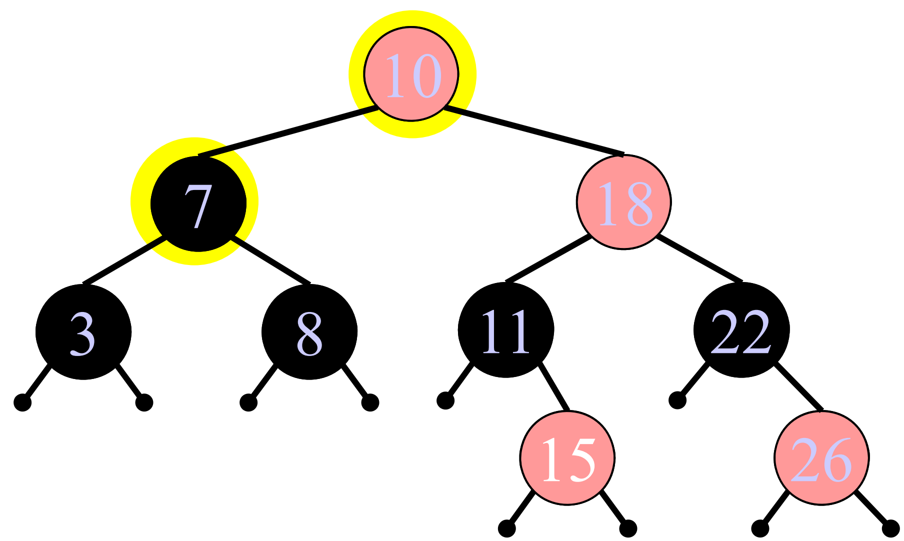

- Insert $x = 15$.
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- RIGHT-ROTATE(18).
- LEFT-ROTATE(7) and recolor.

Insertion into a red-black tree

IDEA: Insert x in tree. Color x red. Only red-black property 4 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

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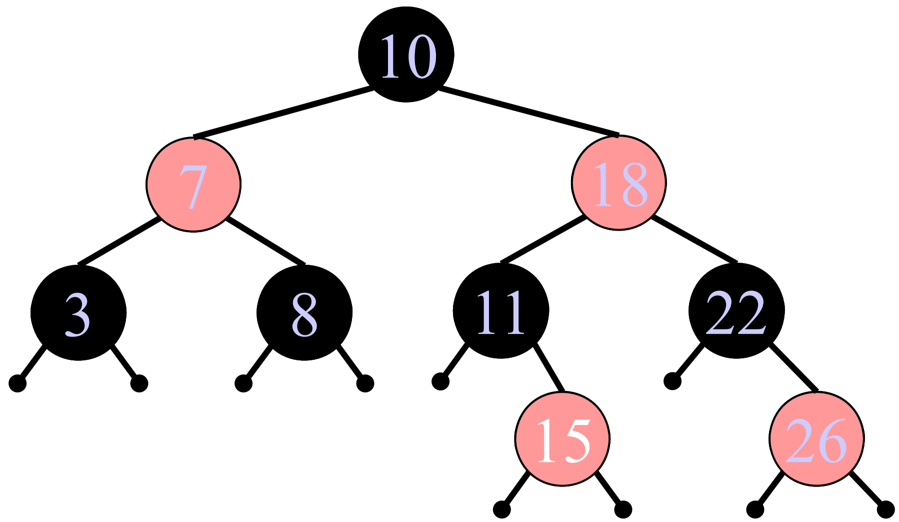
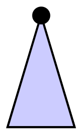
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7) and recolor.

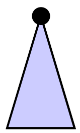


Pseudocode

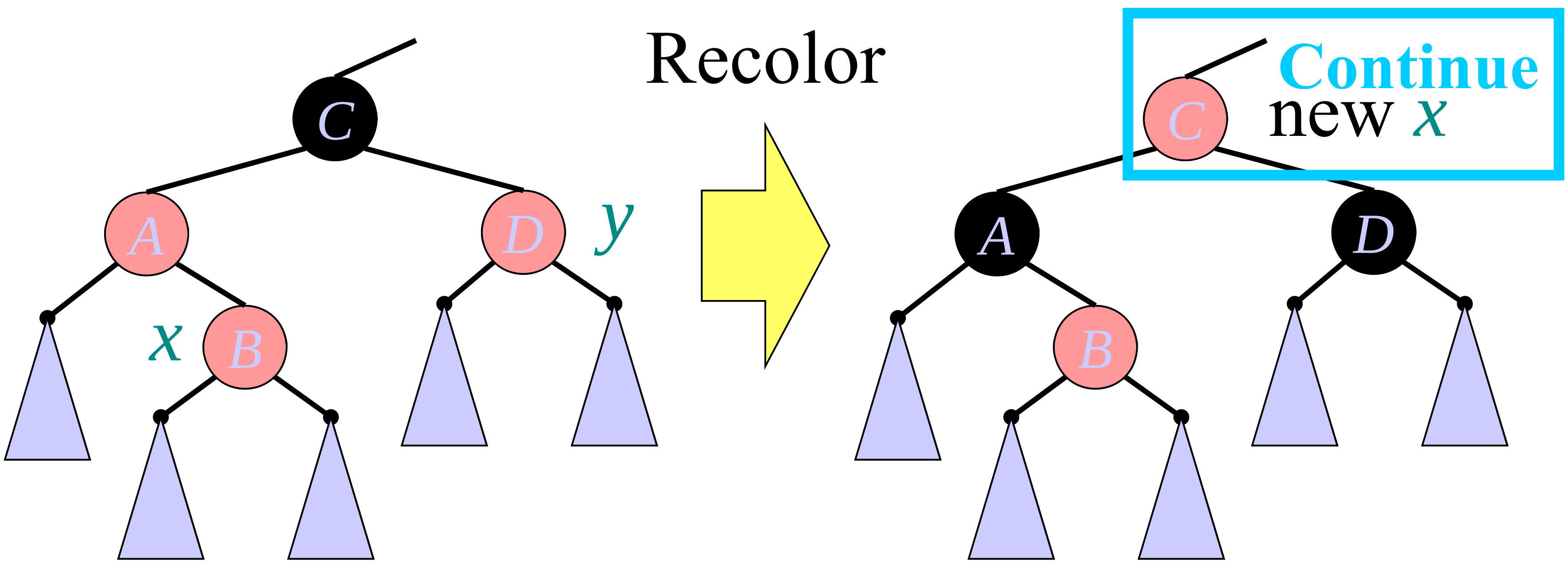
```
RB-INSERT(T, x)
    TREE-INSERT(T, x)
    color[x] ← RED      ⊳ only RB property 4 can be violated
    while x ≠ root[T] and color[p[x]] = RED
        do if p[x] = left[p[p[x]]]
            then y ← right[p[p[x]]]      ⊳ y = aunt/uncle of x
                if color[y] = RED
                 then ⟨Case 1⟩
                 else  if x = right[p[x]]
                         then ⟨Case 2⟩    ⊳ Case 2 falls into Case 3
                        ⟨Case 3⟩
            else ⟨"then" clause with "left" and "right" swapped⟩
    color[root[T]] ← BLACK
```

Graphical notation

Let  denote a subtree with a black root.

All ’s have the same black-height.

Case 1



(Or, *A*'s children are swapped.)

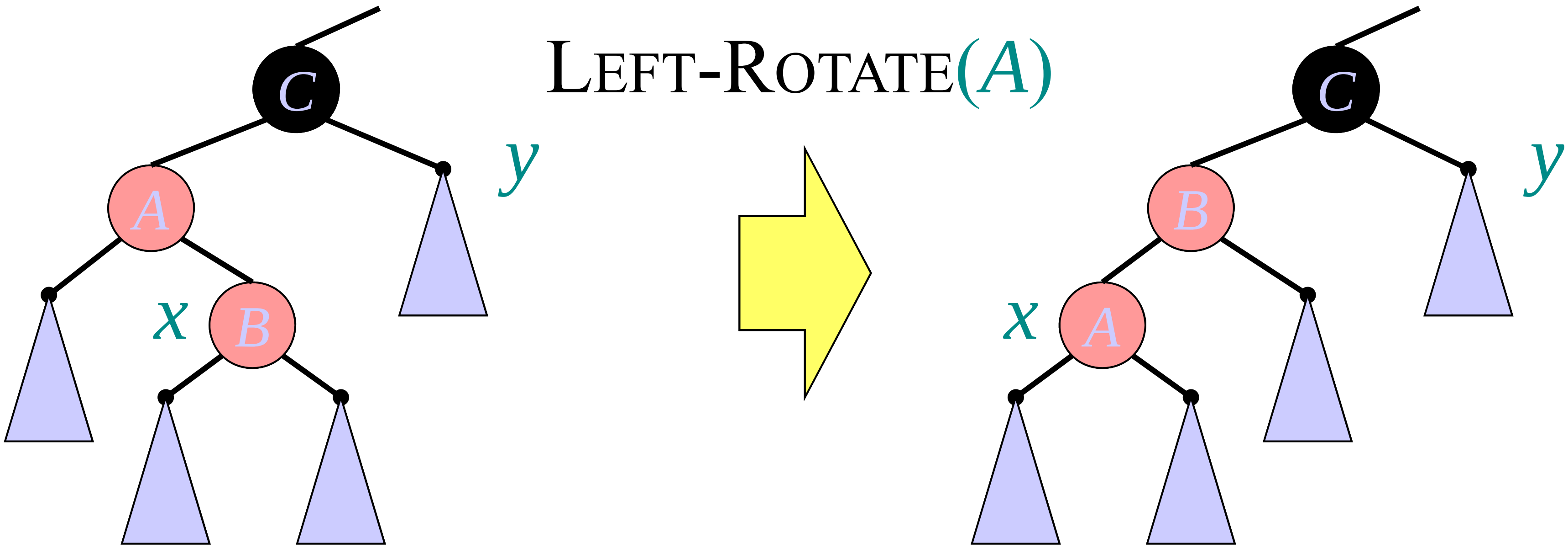
$p[x] = left[p[p[x]]$

$y \leftarrow right[p[p[x]]$

$color[y] = \text{RED}$

Push *C*'s black onto *A* and *D*, and recurse, since *C*'s parent may be red.

Case 2



$p[x] = left[p[p[x]]$

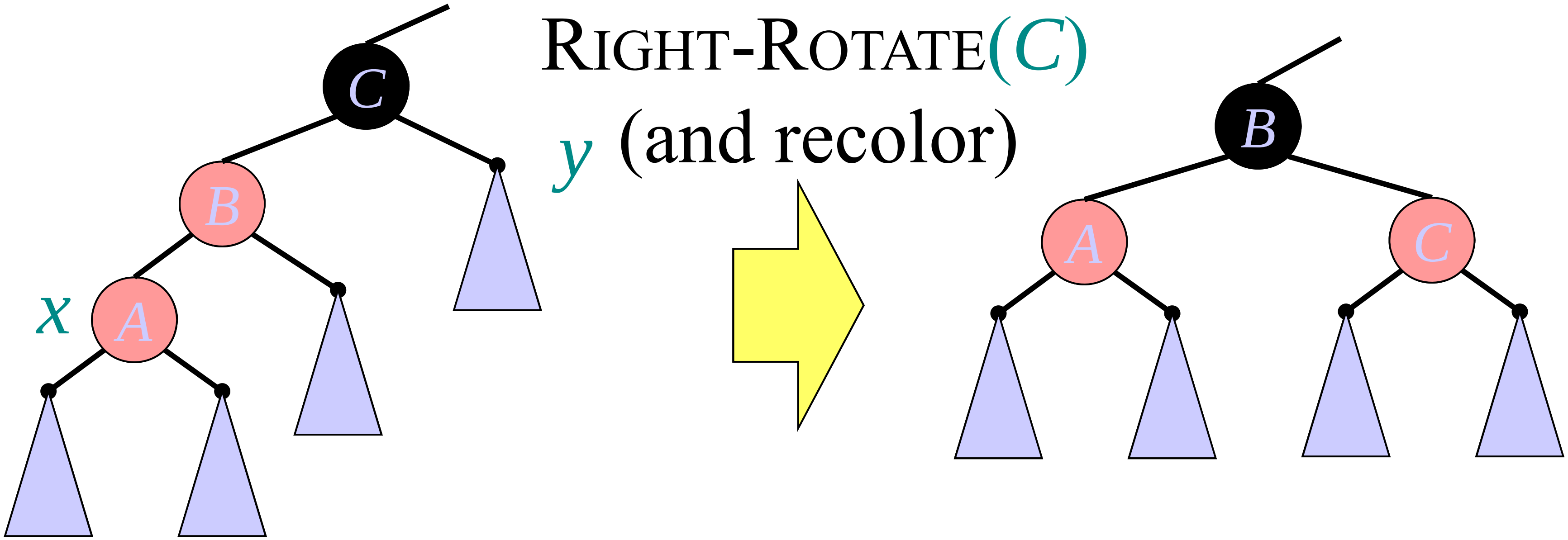
$y \leftarrow right[p[p[x]]$

$color[y] = \text{BLACK}$

$x = right[p[x]]$

Transform to Case 3.

Case 3



$p[x] = left[p[p[x]]$

$y \leftarrow right[p[p[x]]$

$color[y] = \text{BLACK}$

$x = left[p[x]]$

Done! No more violations of RB property 4 are possible.

Analysis

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

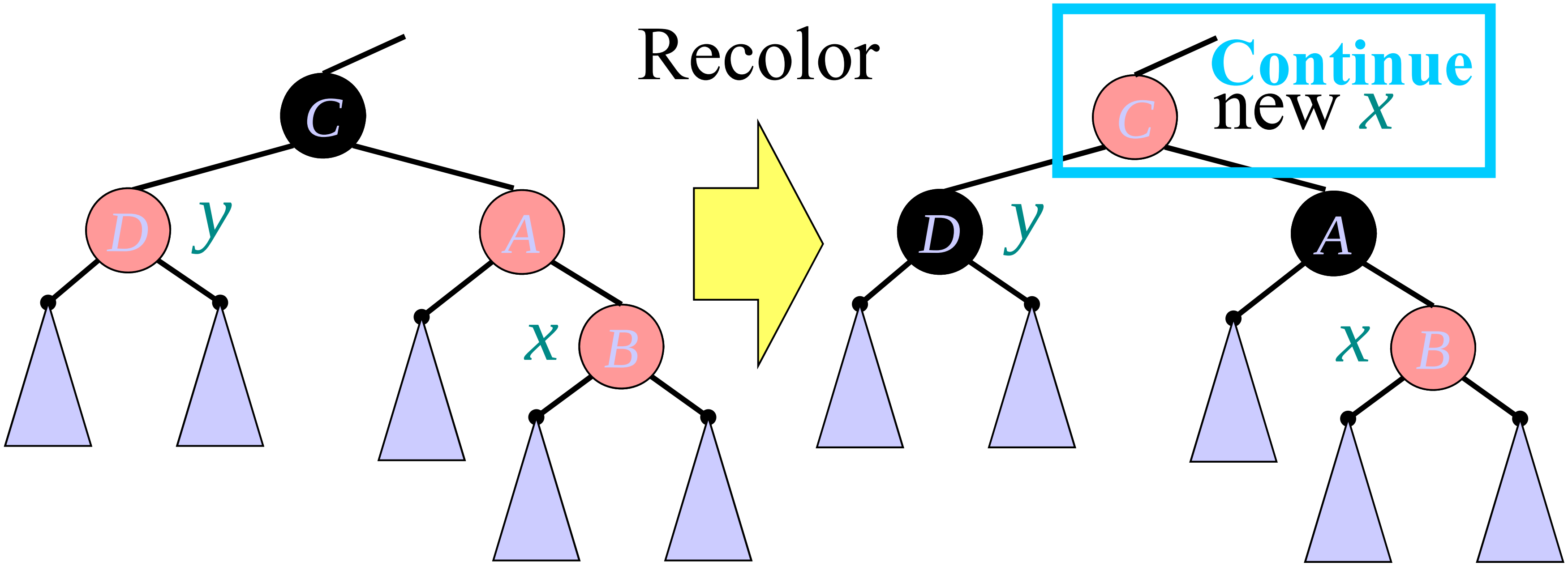
Running time: $O(\log n)$ with $O(1)$ rotations.

RB-DELETE — same asymptotic running time and number of rotations as RB-INSERT.

Pseudocode (part II)

else ⟨**"then"** clause with "*left*" and "*right*" swapped⟩
▻ $p[x] = right[p[p[x]]$
then $y \leftarrow left[p[p[x]]$ ▻ y = aunt/uncle of x
  if $color[y] = \text{RED}$
   then ⟨**Case 1'**⟩
   else if $x = left[p[x]]$
     then ⟨**Case 2'**⟩ ▻ Case 2' falls into Case 3'
    ⟨**Case 3'**⟩
$color[root[T]] \leftarrow \text{BLACK}$

Case 1’



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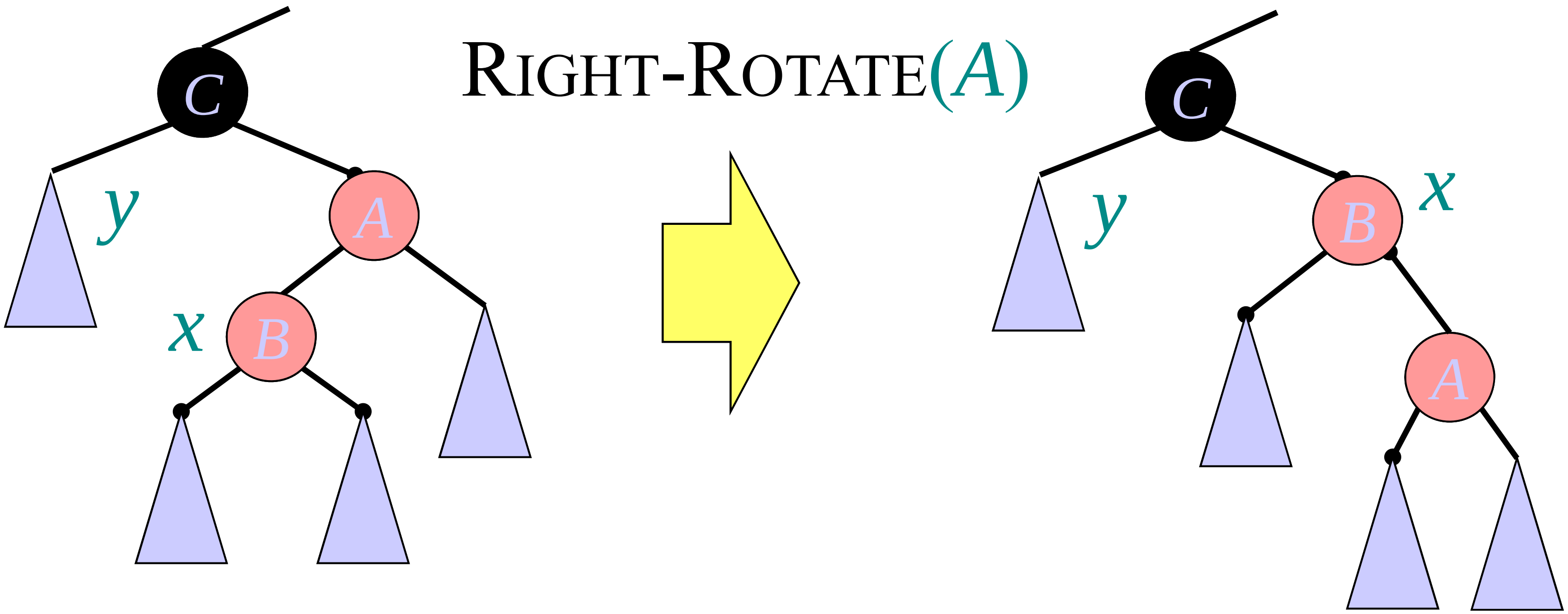
Push C’s black onto A and D, and recurse, since C’s parent may be red.

Case 2'



Transform to Case 3'.

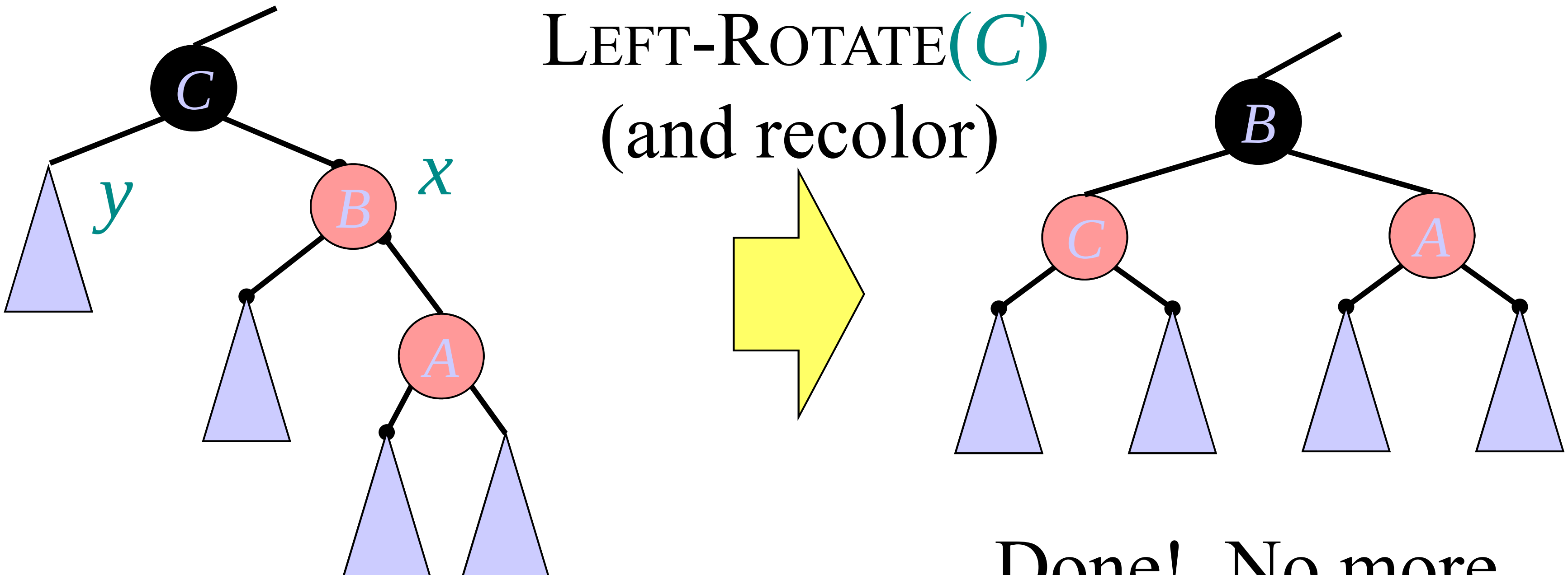
$p[x] = right[p[p[x]]$

$y \leftarrow left[p[p[x]]$

$color[y] = \text{BLACK}$

$x = left[p[x]]$

Case 3’



$p[x] = right[p[p[x]]$

$y \leftarrow left[p[p[x]]$

$color[y] = \text{BLACK}$

$x = right[p[x]]$

Done! No more violations of RB property 4 are possible.