** This homework assignment is required for graduate students only.

1. (5 points) **(Statistical Multiplexing and the Chernoff bound)**

   Consider a link with bandwidth 10 Mbps, which is shared by \( n = 500 \) data sources. At any time, a source is active with a probability of 0.1, and the sources become active independently.

   (a) If a data source transmits at a rate of 100 kbps when active, what is the overflow probability? Using the Chernoff bound to estimate the overflow probability.

   (b) If all the active sources get an equal fraction of the link rate, what is the average throughput per active source?

2. (10 points) **(The Chernoff bound for non-identical Bernoulli random variables)**

   Suppose that \( X_i \) is a Bernoulli random variable with parameter \( \mu_i \) and that \( X_1, X_2, ... \) are independent. In other words, the random variables are independent but not identical. Prove that for \( x > 0 \),

   \[
   \Pr\left( \frac{X_1 + \ldots + X_n}{n} \geq \mu + x \right) \leq e^{-nD((\mu + x)\parallel \mu)},
   \]

   where \( \mu = \frac{1}{n} \sum_{i=1}^{n} \mu_i \).

   Hint: Use the AM-GM (arithmetic mean-geometric mean) inequality:

   \[
   \left( \prod_{i=1}^{n} x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i
   \]

   for non-negative \( x_1, x_2, ..., x_n \).

3. (10 points) **(Data Centers, Backhoes, and Bugs)**

   Data centers alternate between “working” and “down.” There are many reasons why data centers can be down, but for the purpose of this problem we mention only two reasons: (i) a backhoe accidentally dug up some cable, or (ii) a software bug crashed the machines. Suppose that a data center that is working today will be down tomorrow due to backhoe reasons with probability \( \frac{1}{6} \), but will be down tomorrow due to a software bug with probability \( \frac{1}{4} \). A data center that is down today due to backhoe reasons will be up tomorrow with probability \( \frac{3}{4} \). A data center that is down today due to a software bug will be up tomorrow with probability \( \frac{3}{4} \).

   (a) Draw a DTMC for this problem.
(b) Show that the DTMC is irreducible and aperiodic.

(c) What fraction of time is the data center working? (Hint: find the stationary distribution using local balance equations.)

4. (10 points) \textbf{(Geo/D/1 queues)}

Consider a single-server queue with infinite buffer. In each time slot, half a packet is served. In other words, it takes two time slots to serve a packet. Packets arrive to this queue according to an \textit{i.i.d.} Bernoulli process with parameter \( \lambda \). This queue is called the Geo/D/1 queue. Clearly identify the conditions under which the Markov chain has a stationary distribution. Compute the stationary distribution and the average queue length and the average waiting time in steady-state.

Hint: Let \( X(t) \) denote the number of time slots of service remaining. Show that \( X(t) \) is a DTMC that is irreducible and aperiodic. Find the stationary distribution using local balance equations. To find the average queue length, note that the queue length in time slot \( t \) is equal to \( \lceil X(t)/2 \rceil \).