1. (6 points) Use strong induction to prove that every positive integer $n$ can be written as a sum of distinct powers of two. For instance, $19 = 16 + 2 + 1 = 2^4 + 2^1 + 2^0$. (Hint: in the inductive step, separately consider the case where $k + 1$ is even and where it is odd.)

2. (6 points) Give a recursive definition of:
   (a) the sequence $\{a_n\}_{n \in \mathbb{N}}$ where $a_n = 4n - 2$.
   (b) the set of odd positive integers.
   (c) the function $f(n) : \mathbb{Z}^+ \to \mathbb{Z}^+$ where $f(n)$ is the sum of positive integers less than or equal to $n$.

3. (4 points) Let $f_n$ denote the $n$th Fibonacci number. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_nf_{n+1}$ for any positive integer $n$.

4. (4 points) Let $S$ be the subset of the set of ordered pairs of integers defined recursively by
   (a) Base step: $(0, 0) \in S$.
   (b) Recursive step: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

   Use structural induction to show that $5|(a + b)$ for any $(a, b) \in S$.

5. (6 points) Consider climbing a ladder with $n$ rungs. The rungs are spaced such that you can climb one rung or two rungs at a time. Let $r(n)$ be the number of different ways to climb a ladder with $n$ rungs. For example, $r(2) = 2$ because one can climb a 2-rung ladder either as 1 + 1 rungs or as 2 rungs, which are two different climbing patterns.

   (a) (2 points) Give the values of $r(1)$, $r(3)$ and $r(4)$. Justify your answer.
   (b) (4 points) Find a recursive formula for $r(n)$ where $n \geq 1$. Explain your answer.

6. (6 points) Solve the linear recurrences below.
   (a) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$.
   (b) $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$. 