1. **Propositions (6 points)**

Express the following system specifications using the propositions $p$ “The user enters a valid password,” $q$ “Access is granted,” and $r$ “The user has paid the subscription fee” and logical connectives.

(a) “The user has paid the subscription fee, but does not enter a valid password.”

(b) “Access is granted whenever the user has paid the subscription fee and enters a valid password.”

(c) “Access is denied if the user has not paid the subscription fee.”

(d) “If the user has not entered a valid password but has paid the subscription fee, then access is granted.”

Are these system specifications consistent? Justify your answer shortly.

2. **Tautology (4 points)**

Show that $(p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r)$ is a tautology.

3. **Equivalences (6 points)**

Show that $(p \leftrightarrow q)$ and $(p \land q) \lor (\neg p \land \neg q)$ are logically equivalent.

(a) By using truth tables.

(b) By establishing a sequence of equivalences. You may use all equivalences in Table 6 and the first two equivalences in Table 7. Show each step.

4. **NAND (6 points)**

We showed in class that $\{\land, \lor, \neg\}$ is functionally complete, i.e., any Boolean function can be expressed using a combination of $\land, \lor, \neg$.

(a) (2 points) Show that $\{\lor, \neg\}$ is functionally complete.

(b) (4 points) Consider the NAND operator $(\uparrow)$ with the following truth table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \uparrow y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Show that $\{\uparrow\}$ is functionally complete.

5. **Boolean Functions (6 points)**

Find the disjunctive normal forms (also called sum-of-products expansions) of the following Boolean functions (you can use either truth tables or Boolean identities).
(a) $F(x, y) = \neg x \lor y$.
(b) $F(x, y, z) = 1$ if and only if $x \lor y = 0$.

6. **Quantifiers (4 points)** Justify your answers shortly.

   (a) Let $Q(x)$ be the statement “$x + 1 > 2x$”, and let the domain be all integers. Determine the truth value of each of the following statements.
   
   (i) $Q(-1)$  
   (ii) $\forall x : Q(x)$  
   (iii) $\exists x : \neg Q(x)$

   (b) Find a counterexample to each of the following statements where the domain is all real numbers.
   (i) $\forall x : x^2 \neq x$  
   (ii) $\forall x : x^2 \neq 2$  
   (iii) $\forall x : |x| > 0$