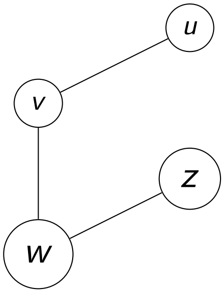
Neighbours

Definition

Two vertices u and v in an undirected graph $G = (V, E)$ are called *adjacent* (or neighbors) in G if $\{u, v\} \in E$. In other words, u and v are neighbors if there is an edge e which has u and v as its endpoints. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Example



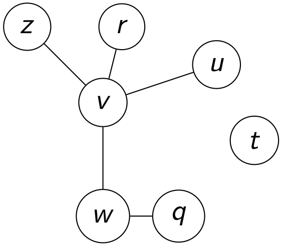
- The neighbour of u is v
- The neighbour of z is w
- The neighbours of w are v and z
- The neighbours of v are w and u

Neighbourhood

Definition

The *neighbourhood* of a vertex v in an undirected graph $G = (V, E)$ is the set of all neighbours of v, which we will denote by $N(v)$. If $A \subseteq V$, then we define $N(A) = \bigcup_{v \in A} N(v)$ to be the set of all neighbours of some vertex in A.

Example



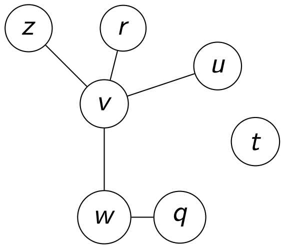
- $N(v) = \{z, r, u, w\}$
- $N(\{r, w\}) = \{v, q\}$
- $N(\{z, r, u\}) = \{v\}$
- $N(\{t\}) = \emptyset$

Degree

Definition

The *degree* of a vertex v, denoted $deg(v)$, in an undirected graph is the number of edges incident with it.

Example



- $deg(t) = 0$
- $deg(u) = 1$
- $deg(r) = 1$
- $deg(z) = 1$
- $deg(q) = 1$
- $deg(w) = 2$
- $deg(v) = 4$

Degree

Theorem

Let $G = (V, E)$ be an undirected graph with m edges. Then:

$$\sum_{v \in V} deg(v) = 2m$$

Proof.

By induction on the number of edges. Alternatively, there is also a simple combinatorial proof – every edge is incident to exactly two vertices and each edge will be counted twice in the sum. □

Example

A graph has 8 vertices with degree 4 and 2 vertices with degree 2. How many edges does this graph have?

Degree

Theorem

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Example

A graph has 8 vertices with degree 4 and 2 vertices with degree 2. How many edges does this graph have? Using the theorem, we see

$$\sum_{v \in V} deg(v) = 8 \times 4 + 2 \times 2 = 36.$$

Thus, the graph must have 36/2=18 edges.

Remark: there are many different graphs which satisfy these conditions.

Degree

Theorem

Let $G = (V, E)$ be an undirected graph. Then G has an even number of vertices of odd degree.

Proof.

Let V_1 be the set of vertices with odd degree and let V_2 be the set of vertices with even degree. Then:

$$2|E| = \sum_{v \in V} deg(v) = (\sum_{v \in V_1} deg(v)) + (\sum_{v \in V_2} deg(v))$$

However, it should be clear that 2 divides $\sum_{v \in V_2} deg(v)$ which implies that 2 must also divide $\sum_{v \in V_1} deg(v)$. □

Complete Graphs

Definition

For $n \in \mathbb{N}$, the *complete graph* on n vertices, denoted K_n, is the undirected graph which contains n vertices and exactly one edge between any distinct pair of vertices.

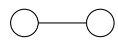
Example

K_0 :

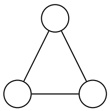
(empty graph)

K_1 :

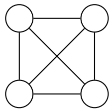
K_2 :



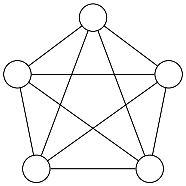
K_3 :



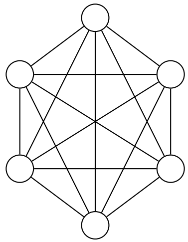
K_4 :



K_5 :



K_6 :

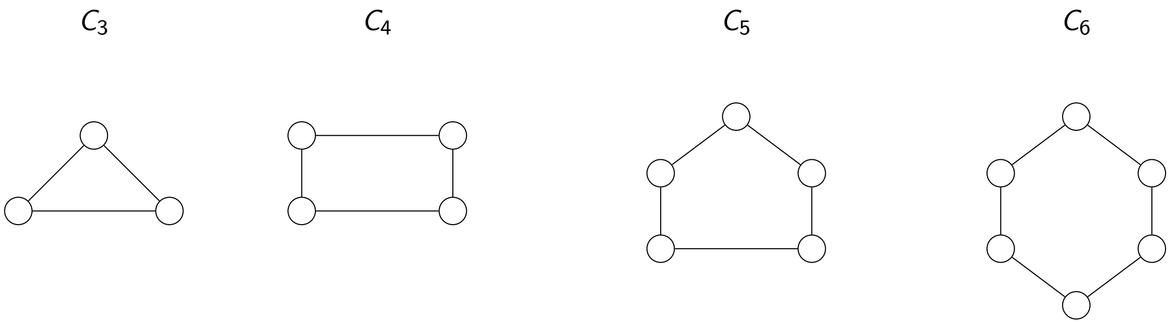


Cycles

Definition

For $n \geq 3$, the *cycle graph* on n vertices, denoted C_n, is the undirected graph which contains n vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$

Example



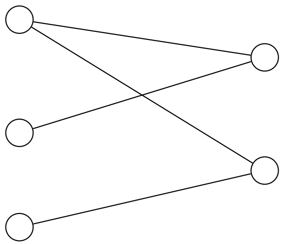
Remark: Our definition of undirected graph does not allow us to define C_2, but we may do it using pseudographs. Why?

Bipartite graphs

Definition

An undirected graph $G = (V, E)$ is called *bipartite* iff V can be partitioned into two disjoint nonempty sets V_1 and V_2, such that every edge in E is incident to one vertex from V_1 and one vertex from V_2.

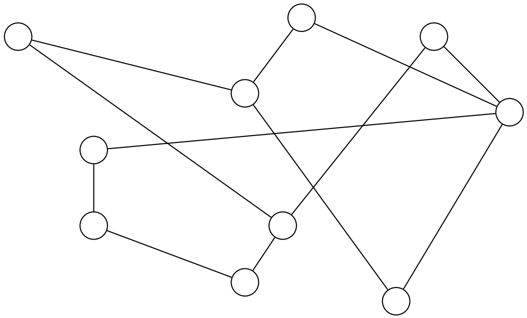
Example



K_2 is bipartite, but K_n is not bipartite for $n \neq 2$. Why?
C_n for $n \geq 3$ is bipartite iff n is even. Why?

Bipartite graphs (continued)

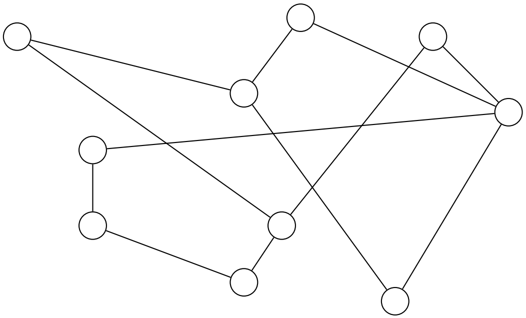
Question: How can we determine if the following graph is bipartite?



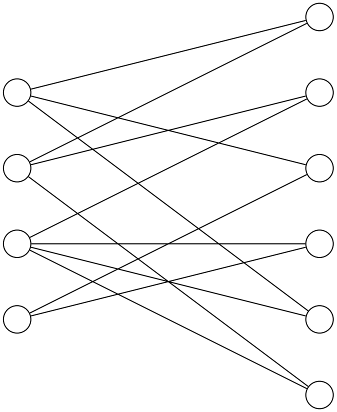
Try to rearrange the vertices in two clusters such that there are no edges between any two vertices in the same cluster.

Bipartite graphs (continued)

Question: How can we determine if the following graph is bipartite?



Try to rearrange the vertices in two clusters such that there are no edges between any two vertices in the same cluster.
After rearranging, the above graph becomes:



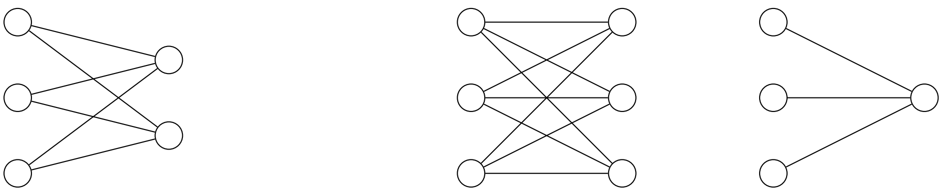
Complete bipartite graphs

Definition

Given $n, m \in \mathbb{N}_+$, we denote with $K_{m,n}$ the *complete bipartite graph* on n and m vertices which is defined to be the bipartite graph whose vertex set is partitioned into two subsets - one on n vertices and the other on m vertices, where each of the n vertices from the first partition is adjacent to all m vertices of the second partition.

Example

$K_{3,2}$ $K_{3,3}$ $K_{1,3}$



Remark 1: $\forall n, m \in \mathbb{N}_+ : K_{n,m} = K_{m,n}$
Remark 2: $K_{0,m}$ does not exist, because we require partitions of the vertex set to be nonempty.
Question: In general $K_{m,n}$ is not complete. For what values of n and m is $K_{m,n}$ complete?

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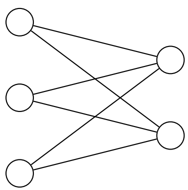
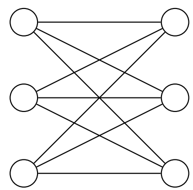
Example

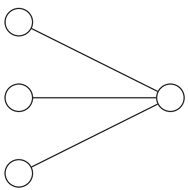
$K_{3,2}$



$K_{3,3}$



$K_{1,3}$



Remark 1: $\forall n, m \in \mathbb{N}_+ : K_{n,m} = K_{m,n}$
Remark 2: $K_{0,m}$ does not exist, because we require partitions of the vertex set to be nonempty.
Question: In general $K_{m,n}$ is not complete. For what values of n and m is $K_{m,n}$ complete?
Answer: $K_{1,1} = K_2$. If $n \neq 1$ and $m \neq 1$, then $K_{m,n}$ is not complete.