Graphs

March 24 2017

Motivation

- A graph is a simple mathematical structure which consists of a set of *vertices* that are related to each other via *edges*.
- They admit a simple and intuitive graphical presentation which has lead to many applications in multiple fields.
- Graphs are important mathematical objects with applications in knot theory, category theory, geometry and others.
- Arguably the most important data structure in computer science, where they can be used to represent networks, processes, have multiple applications in compiler optimization and can even be used for reasoning about quantum information processing, among other applications.
- Multiple other applications in fields ranging from sociology to chemistry, biology and physics.

Examples

Figure: Some random social network graph that I found on Wolfram.

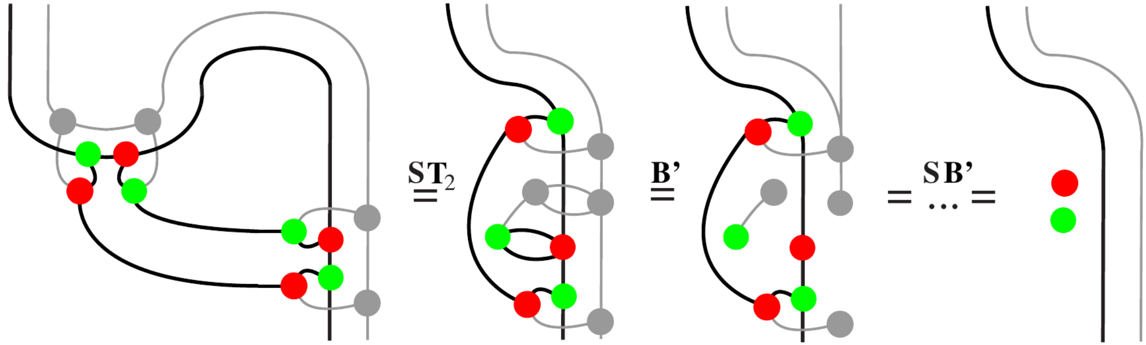


Figure: B. Coecke, R. Duncan (2011). Diagrammatic representation of quantum teleportation.

More Examples

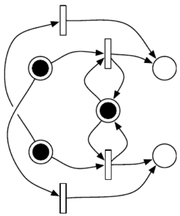


Figure: P. Sobocinski (2010). Petri nets have applications in concurrenncy.

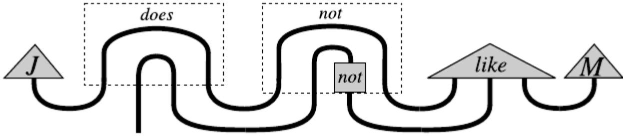


Figure: B. Coecke, E. Grefenstette, M. Sadrzadeh (2013). Applications in compositional semantics for computational linguistics.

History

- Leonard Euler (1707–1783) is widely regarded as the first person to initiate the study of graph theory.
- One of the most prolific mathematicians of all time
- Königsberg, Prussia (now Kaliningrad, Russian exclave) had a popular riddle associated to it, called the "Seven Bridges of Königsberg".
- The problem was how to cross each of the seven bridges while crossing each bridge exactly once.
- Euler knew that algebra, geometry, combinatorics and all other contemporary fields of mathematics could not answer the question.
- So he initiated graph theory and proved the negative resolution of the problem.

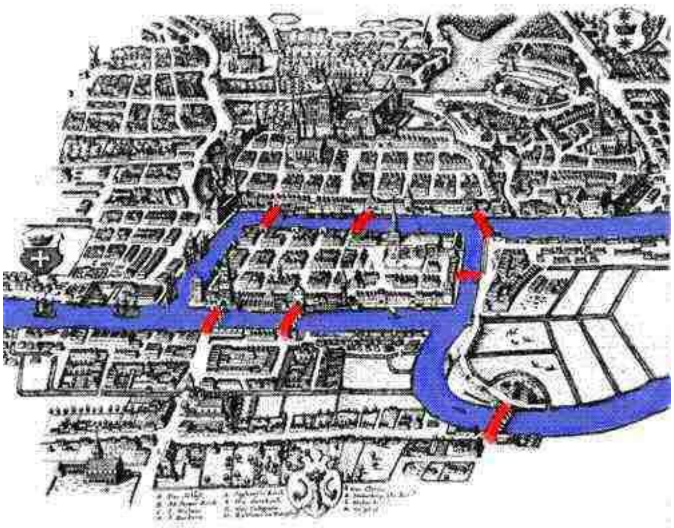


Figure: http://www-history.mcs.st-andrews.ac.uk/Extras/Konigsberg.html

Pseudographs

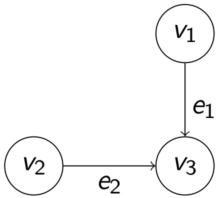
Definition

A *(directed) pseudograph* $G = (V, E, s, t)$ is a set of *vertices* V, together with a set of *edges* E and two functions $s, t : E \to V$ called the *source* and *target* functions.

Example

Setting $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2\}$ and $s(e_1) = v_1, s(e_2) = v_2, t(e_1) = v_3, t(e_2) = v_3$, then the tuple (V, E, s, t) is a pseudograph.

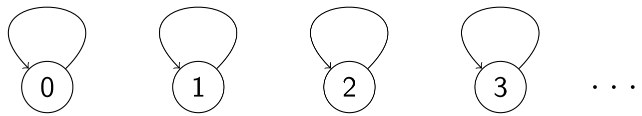
- Pseudographs admit a nice graphical notaiton. Vertices can be depicted as small circular nodes and the edges can be depicted as directed arrows connecting them. The vertices may be placed in any order. For each edge e_i we draw an arrow with *source* $s(e_i)$ and *target* $t(e_i)$.



Examples

Example

Every set A may be seen as a pseudograph. Define $V = E = A$ and set $s(a) = t(a) = a$. Then, the graphical depiction of $\mathbb{N}$ as a pseudograph is given by:



Definition

If an edge e is such that $s(e) = t(e)$ in a pseudograph, then e is called a *self-loop*.

- All of the edges in the above example are self-loops.

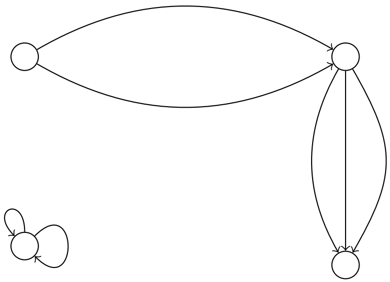
Remark: Pseudographs may be infinite as the above example demonstrates. However, from now on we will consider only graphs and pseudographs where the set of edges and vertices are both finite. For the remainder of the course, we will assume that all of our graphs and pseudographs are finite unless otherwise noted.

More examples

Definition

If a pseudograh G has edges e_1, e_2, such that $s(e_1) = s(e_2)$ and $t(e_1) = t(e_2)$, then the edges e_1 and e_2 are called *parallel edges* and the pseudograph G is said to have *multiple edges*.

Example



Graphs

- For many applications parallel edges and self-loops are not required.
- A pseudograph who has neither is simply called a (directed) graph.

Definition

A *(directed) graph* is a pseudograph which has no pair of parallel edges and no self-loops.

- Compared to pseudographs, graphs are obviously simpler. This allows us to introduce an *equivalent* definition for graphs which is more concise.

Definition

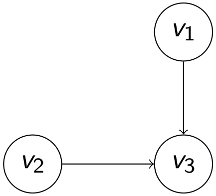
A *(directed) graph* $G = (V, E)$ is a set of *vertices* V together with a relation $E \subseteq V \times V$, s.t. $(v, v) \notin E$ for any vertex $v \in V$.

Remark: The absence of parallel edges allows us to represent edges using only a relation, instead of using a pair of functions. The requirement for self-loops is irrelevant for this presentation, but it is a standard assumption in the graph-theoretic literature.

Example

Example

Consider the graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{(v_1, v_3), (v_2, v_3)\}$. Then, G is the same as the pseudograph from slide 6:



In general, for a graph $G = (V, E)$, each element of the relation is of the form (v_i, v_j). This is graphically represented by drawing an arrow from vertex v_i to vertex v_j. Because (directed) graphs are special kinds of (directed) pseudographs, we will use the same graphical presentation for both.

Question: Why can't we use graphs (and thus relations) to represent pseudographs with parallel edges?

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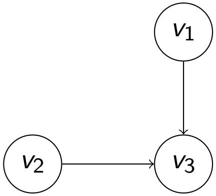


In general, for a graph $G = (V, E)$, each element of the relation is of the form (v_i, v_j). This is graphically represented by drawing an arrow from vertex v_i to vertex v_j. Because (directed) graphs are special kinds of (directed) pseudographs, we will use the same graphical presentation for both.

Question: Why can't we use graphs (and thus relations) to represent pseudographs with parallel edges?

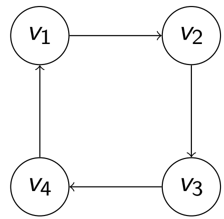
- Relations cannot be used to represent the multiplicity of its elements, because they are simply a subset of $V \times V$.

More Examples

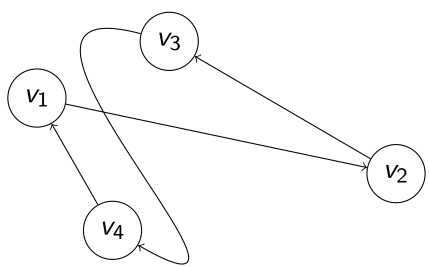
- We will mostly consider graphs (as opposed to pseudographs) in this course. Here are a few more examples.

Example

The graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ is given by:



The only thing which matters in the graphical depiction is the number of vertices and how they are connected via edges to each other. Thus, the following graphical representation:

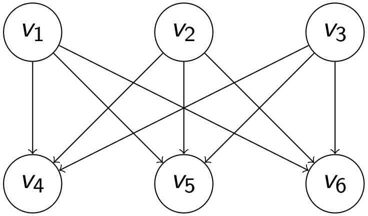


is equivalent to the original one.

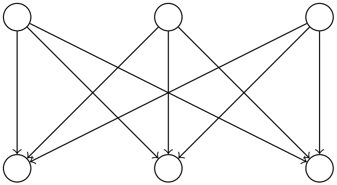
More Examples

Example

Although not required, it is customary to draw graphs in a way where the edges do not intersect, if possible. This usually makes the presentation more clear. However, this is not always possible.



The above graph can never be drawn in a such way (try it out). The proof of this fact is complicated. It's set-theoretic presentation can be given by: $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and
$E = \{(v_1, v_4), (v_1, v_5), (v_1, v_6), (v_2, v_4), (v_2, v_5), (v_2, v_6), (v_3, v_4), (v_3, v_5), (v_3, v_6)\}$.
If the details of the set-theoretic presentation are irrelevant for our purposes, then we may simply draw it:



Directed vs Undirected (pseudo)graphs

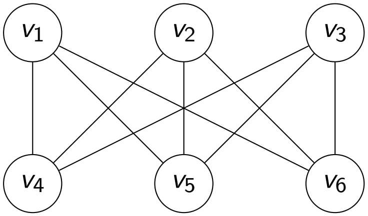
- A major feature of a (pseudo)graph is whether its edges are directed or not
- Directed (pseudo)graphs are more general than undirected ones
- We will only formally define undirected graphs because we won't be talking much about undirected pseudographs

Definition

An *undirected graph* is a pair $G = (V, E)$, where V is a set of vertices and E is a set of 2-element subsets of V.

Example

The undirected version of the last graph is given set-theoretically by: $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}\}$.



The main idea is to replace *ordered* pairs (u, v) in the directed case with *unordered* pairs $\{u, v\}$ in the undirected case. Thus, edges in an undirected graph have no information about direction and are graphically depicted as wires (as opposed to arrows in the directed case).

Types of graphs

- So far we have seen several different types of graphs
- We have pseudographs vs graphs
- Each of those 2 can be directed or undirected
- Each of those 4 has multiple equivalent definitions
- In addition we may also define pseudographs without self-loops which are usually referred to as multigraphs
- We may also define graphs with self-loops but no parallel edges
- To make matters even more complicated, there is no standard notion of what "graph" means in the literature – different authors use different *nonequivalent* definitions for what they are talking about
- Because of all these reasons, you need to be very careful when consulting graph-theoretic literature.

Seven Bridges of Königsberg

- Going back to the problem that started graph theory

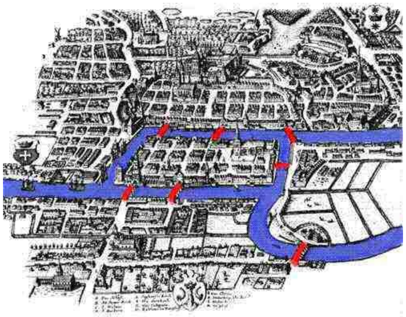
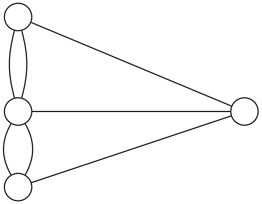


Figure: http://www-history.mcs.st-andrews.ac.uk/Extras/Konigsberg.html

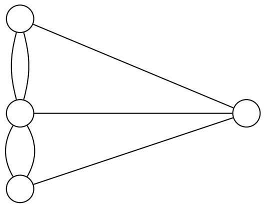
- Size of the landmasses is irrelevant, therefore contract to a single point (vertex)
- The geometry of the bridges is irrelevant – we only care which bridge connects which two landmasses, therefore represent bridges as undirected edges.
- Thus we end up with an undirected pseudograph



Seven Bridges of Königsberg (continued)

- So, the problem may be reformulated in graph-theoretic terms.

Problem: Given the following undirected pseudograph:



does there exist a path which traverses all edges of the pseudograph exactly once?

- In the lectures which follow, we will introduce enough terminology to make the statement of this problem mathematically precise and we will eventually prove that this is impossible.