Graphs

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A graph is a simple mathematical structure which consists of a set of vertices that are related to each other via edges.

They admit a simple and intuitive graphical presentation which has lead to many applications in multiple fields.

Graphs are important mathematical objects with applications in knot theory, category theory, geometry and others.

Arguably the most important data structure in computer science, where they can be used to represent networks, processes, have multiple applications in compiler optimization and can even be used for reasoning about quantum information processing, among other applications.

Multiple other applications in fields ranging from sociology to chemistry, biology and physics.
Examples

Figure: Some random social network graph that I found on Wolfram.

Figure: B. Coecke, R. Duncan (2011). Diagrammatic representation of quantum teleportation.
More Examples

**Figure**: P. Sobocinski (2010). Petri nets have applications in concurrency.

**Figure**: B. Coecke, E. Grefenstette, M. Sadrzadeh (2013). Applications in compositional semantics for computational linguistics.
History

- Leonard Euler (1707–1783) is widely regarded as the first person to initiate the study of graph theory.
- One of the most prolific mathematicians of all time
- Königsberg, Prussia (now Kaliningrad, Russian exclave) had a popular riddle associated to it, called the “Seven Bridges of Königsberg”.
- The problem was how to cross each of the seven bridges while crossing each bridge exactly once.
- Euler knew that algebra, geometry, combinatorics and all other contemporary fields of mathematics could not answer the question.
- So he initiated graph theory and proved the negative resolution of the problem.

Figure: http://www-history.mcs.st-andrews.ac.uk/Extras/Konigsberg.html
**Definition**

A *(directed) pseudograph* \( G = (V, E, s, t) \) is a set of vertices \( V \), together with a set of edges \( E \) and two functions \( s, t : E \to V \) called the *source* and *target* functions.

**Example**

Setting \( V = \{v_1, v_2, v_3\} \), \( E = \{e_1, e_2\} \) and \( s(e_1) = v_1, s(e_2) = v_2, t(e_1) = v_3, t(e_2) = v_3 \), then the tuple \( (V, E, s, t) \) is a pseudograph.

- Pseudographs admit a nice graphical notation. Vertices can be depicted as small circular nodes and the edges can be depicted as directed arrows connecting them. The vertices may be placed in any order. For each edge \( e_i \) we draw an arrow with *source* \( s(e_i) \) and *target* \( t(e_i) \).
Example

Every set $A$ may be seen as a pseudograph. Define $V = E = A$ and set $s(a) = t(a) = a$. Then, the graphical depiction of $\mathbb{N}$ as a pseudograph is given by:

![Graphical depiction of $\mathbb{N}$ as a pseudograph](image)

Definition

If an edge $e$ is such that $s(e) = t(e)$ in a pseudograph, then $e$ is called a *self-loop*.

- All of the edges in the above example are self-loops.

Remark: Pseudographs may be infinite as the above example demonstrates. However, from now on we will consider only graphs and pseudographs where the set of edges and vertices are both finite. For the remainder of the course, we will assume that all of our graphs and pseudographs are finite unless otherwise noted.
**Definition**

If a pseudograph $G$ has edges $e_1, e_2$, such that $s(e_1) = s(e_2)$ and $t(e_1) = t(e_2)$, then the edges $e_1$ and $e_2$ are called *parallel edges* and the pseudograph $G$ is said to have *multiple edges*.

**Example**
For many applications parallel edges and self-loops are not required.

A pseudograph who has neither is simply called a (directed) graph.

Definition
A (directed) graph is a pseudograph which has no pair of parallel edges and no self-loops.

Compared to pseudographs, graphs are obviously simpler. This allows us to introduce an equivalent definition for graphs which is more concise.

Definition
A (directed) graph $G = (V, E)$ is a set of vertices $V$ together with a relation $E \subseteq V \times V$, s.t. $(v, v) \notin E$ for any vertex $v \in V$.

Remark: The absence of parallel edges allows us to represent edges using only a relation, instead of using a pair of functions. The requirement for self-loops is irrelevant for this presentation, but it is a standard assumption in the graph-theoretic literature.
Example

Consider the graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{(v_1, v_3), (v_2, v_3)\}$. Then, $G$ is the same as the pseudograph from slide 6:

![Graph Diagram]

In general, for a graph $G = (V, E)$, each element of the relation is of the form $(v_i, v_j)$. This is graphically represented by drawing an arrow from vertex $v_i$ to vertex $v_j$. Because (directed) graphs are special kinds of (directed) pseudographs, we will use the same graphical presentation for both.

**Question:** Why can’t we use graphs (and thus relations) to represent pseudographs with parallel edges?
Example

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**Question:** Why can’t we use graphs (and thus relations) to represent pseudographs with parallel edges?

- Relations cannot be used to represent the multiplicity of its elements, because they are simply a subset of $V \times V$. 
More Examples

- We will mostly consider graphs (as opposed to pseudographs) in this course. Here are a few more examples.

Example
The graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ is given by:

![Graph Diagram](image)

The only thing which matters in the graphical depiction is the number of vertices and how they are connected via edges to each other. Thus, the following graphical representation:

![Graph Diagram](image)

is equivalent to the original one.
Example

Although not required, it is customary to draw graphs in a way where the edges do not intersect, if possible. This usually makes the presentation more clear. However, this is not always possible.

The above graph can never be drawn in such a way (try it out). The proof of this fact is complicated. It's set-theoretic presentation can be given by: \( V = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) and \( E = \{(v_1, v_4), (v_1, v_5), (v_1, v_6), (v_2, v_4), (v_2, v_5), (v_2, v_6), (v_3, v_4), (v_3, v_5), (v_3, v_6)\} \). If the details of the set-theoretic presentation are irrelevant for our purposes, then we may simply draw it:
Directed vs Undirected (pseudo)graphs

- A major feature of a (pseudo)graph is whether its edges are directed or not
- Directed (pseudo)graphs are more general than undirected ones
- We will only formally define undirected graphs because we won’t be talking much about undirected pseudographs

**Definition**

An *undirected graph* is a pair $G = (V, E)$, where $V$ is a set of vertices and $E$ is a set of 2-element subsets of $V$.

**Example**

The undirected version of the last graph is given set-theoretically by: $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}\}$. 

![Graph Diagram]

The main idea is to replace *ordered* pairs $(u, v)$ in the directed case with *unordered* pairs $\{u, v\}$ in the undirected case. Thus, edges in an undirected graph have no information about direction and are graphically depicted as wires (as opposed to arrows in the directed case).
- So far we have seen several different types of graphs
- We have pseudographs vs graphs
- Each of those 2 can be directed or undirected
- Each of those 4 has multiple equivalent definitions
- In addition we may also define pseudographs without self-loops which are usually referred to as multigraphs
- We may also define graphs with self-loops but no parallel edges
- To make matters even more complicated, there is no standard notion of what “graph” means in the literature – different authors use different *nonequivalent* definitions for what they are talking about
- Because of all these reasons, you need to be very careful when consulting graph-theoretic literature.
Seven Bridges of Königsberg

- Going back to the problem that started graph theory

![Image](http://www-history.mcs.st-andrews.ac.uk/Extras/Konigsberg.html)

**Figure:** [http://www-history.mcs.st-andrews.ac.uk/Extras/Konigsberg.html](http://www-history.mcs.st-andrews.ac.uk/Extras/Konigsberg.html)

- Size of the landmasses is irrelevant, therefore contract to a single point (vertex)
- The geometry of the bridges is irrelevant – we only care which bridge connects which two landmasses, therefore represent bridges as undirected edges.
- Thus we end up with an undirected pseudograph
So, the problem may be reformulated in graph-theoretic terms.

**Problem:** Given the following undirected pseudograph:

![Pseudograph Diagram]

does there exist a path which traverses all edges of the pseudograph exactly once?

- In the lectures which follow, we will introduce enough terminology to make the statement of this problem mathematically precise and we will eventually prove that this is impossible.