1. **(Logic and Proof)**

   (a) Prove that $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \exists z \in \mathbb{R} : (x \neq 0) \rightarrow (y \times z > \frac{1}{x})$ (10)

   Solution: For an arbitrary $x \in \mathbb{R}$, such that $x \neq 0$, choose $y = (\frac{1}{x}) + 1$ and choose $z = 1$. Then $y \times z = \frac{1}{x} + 1 > \frac{1}{x}$. □

   (b) Prove that $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \forall z \in \mathbb{R} : (y - z \leq x) \vee (z - y \leq x)$ (10)

   Solution: Set $x = 0$. Then for arbitrary $y, z \in \mathbb{R}$, we have:

$$\begin{aligned} &\text{If } y = z, \text{ then } y - z = 0 = x. \\ &\text{If } y < z, \text{ then } y - z < 0 = x. \\ &\text{If } y > z, \text{ then } z - y < 0 = x. \end{aligned}$$

   Therefore, in all possible cases one of the clauses of the disjunction is true. □

2. **(Set Theory)**
   Let A be a finite set with $n \geq 1$ elements $A = \{A_1, A_2, \ldots, A_n\}$, where each A_i is itself a set, such that for any i, j we have $A_i \subseteq A_j$ or $A_j \subseteq A_i$.

   (a) Prove that A contains an element A_k, such that A_k is not a subset of any other element in A. (10)

   Solution: Assume for contradiction that no such element A_k exists. Then, any A_i is such that it has a *proper* superset A_j for some choice of $j \neq i$. Thus, $A_1 \subset A_{i_1}$ for some $i_1 \neq 1$. But then, also $A_{i_1} \subset A_{i_2}$, for some i_2 which is distinct from 1 and i_1. By iterating this idea, we get:

$$A_1 \subset A_{i_1} \subset A_{i_2} \subset \cdots \subset A_{i_n}$$

   where all of the indices are distinct. However, this sequence contains $n + 1$ different elements of A which is a contradiction, because A has only n elements. □

   (b) Prove that A_k is a superset of every element of A. (10)

   Solution: Choose an arbitrary $i \in \{1, 2, \ldots n\}$. If $i = k$, then clearly $A_k = A_i \subseteq A_k$. Otherwise, $i \neq k$ and thus it must be the case $A_i \neq A_k$. The premise of the problem gives us:

$$A_i \subseteq A_k \text{ or } A_k \subseteq A_i$$

   But, from part (a), we know that the latter is not true. Therefore, it must be the case that $A_i \subset A_k$. Because we have chosen the index i arbitrarily, we conclude that A_k is a superset of all elements in A. □

3. **(Induction and Recursion)** (20)
   Consider the function $f : \mathbb{N} \rightarrow \mathbb{N}$ recursively defined by:

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(2) &= 2 \\ f(n) &= (f(n-3) + f(n-2) + 3)f(n-1) \end{aligned}$$

   Prove by induction that $f(n) \geq n$ for $n \in \mathbb{N}$.

Solution: We will prove this using strong induction.

Base case: If $n = 0$ or $n = 1$ or $n = 2$, we see that the proposition is true.

Step case: Assume that $f(k) \geq k$ for all $k \in \{0, 1, \dots n\}$ for some $n \geq 2$. We will show that $f(n+1) \geq n+1$ which will conclude the proof.

$$
\begin{aligned}
f(n+1) &= (f(n-2) + f(n-1) + 3)f(n) && \text{(Definition)} \\
&\geq (n-2+n-1+3)n && \text{(Induction hypothesis)} \\
&= 2n^2 \\
&= n^2 + n^2 \\
&> n^2 + 1 && (n^2 > 1) \\
&> n + 1 && (n^2 > n)
\end{aligned}
$$

Note, that for the last two inequalities, the assumption that $n \geq 2$ is important. □

4. **(Relations)**
Consider the set $\mathbb{N}_\top = \mathbb{N} \cup \{\top\}$. We define a binary relation $\sqsubseteq$ on $\mathbb{N}_\top$ by:

$$n \sqsubseteq m \text{ iff } m = \top \text{ or } n \leq m$$

where $n \leq m$ is the standard order on $\mathbb{N}$.

(a) Prove that $(\mathbb{N}_\top, \sqsubseteq)$ is a poset. (10)

Solution: We have to show that the relation is reflexive, antisymmetric and transitive.

Reflexivity: If $n \in \mathbb{N}$, then clearly $n \leq n$ and thus $n \sqsubseteq n$. Otherwise $n = \top$ and then by definition $n \sqsubseteq n$.

Antisymmetry: Assume $n \sqsubseteq m$ and $m \sqsubseteq n$. If $m = \top$, then the latter requires $n = \top = m$. If $m \neq \top$, then $m \in \mathbb{N}$ and then $n \sqsubseteq m$ implies $n \in \mathbb{N}$ and $n \leq m$. But then, $m \sqsubseteq n$ implies $m \leq n$ and so we conclude $m = n$, as required.

Transitivity: Assume $n \sqsubseteq m$ and $m \sqsubseteq k$. If $k = \top$, then by definition $n \sqsubseteq k$, as required. If $k \neq \top$, then $k \in \mathbb{N}$ and thus the second hypothesis implies $m \in \mathbb{N}$ with $m \leq k$. Because $m \in \mathbb{N}$, the first hypothesis implies $n \in \mathbb{N}$ with $n \leq m$. Thus, we get $n \leq m \leq k$ and therefore $n \sqsubseteq k$, as required. □

(b) Prove that any non-empty subset of $\mathbb{N}_\top$ has a least upper bound in $\mathbb{N}_\top$. (10)

Solution: Let $X \subseteq \mathbb{N}_\top$ be an arbitrary non-empty subset. Let U be the set of upper bounds of X. In other words:

$$U = \{u \mid u \in \mathbb{N}_\top, x \sqsubseteq u, \text{ for every } x \in X\}$$

We know that U is not empty, because $\top \in U$.

If U has no other elements, then $\top$ is the least upper bound of X by definition.

If U does contain other elements, then clearly $\top$ is not the least upper bound, because $\top$ is the largest element (in both U and $\mathbb{N}_\top$). Then, the set $U' = U - \{\top\}$ is a non-empty subset of $\mathbb{N}$. We know that the natural numbers equipped with the standard order is a *well-ordered* set and therefore U' has a least element $u \in U'$, which is the least upper bound of X, by construction. □

5. **(Graphs)** (20)

Let G be a graph which contains a simple circuit of odd length. Prove that $\chi(G) \geq 3$.

Solution: Assume for contradiction that $\chi(G) \leq 2$. Then, there exists a 2-coloring of G. An arbitrary circuit of odd length has the form:

$$v_1 \to v_2 \to \cdots \to v_{2n+1} \to v_1$$

Because G can be 2-colored, this means we can 2-color this circuit as well. We will use the colors 1 and 2 for this purpose. Without loss of generality, assume that v_1 has color 1. Because v_1 and v_2 are neighbours, this leaves us no choice for v_2 and it must be colored using color 2. But then, v_3 must have color 1, because it and v_2 are neighbours. By iterating this argument, we see that v_{2k+1} has color 1 and v_{2k} has color 2, for any choice of k. But then we get a contradiction, because v_{2n+1} and v_1 have the same color and they are adjacent.

Therefore, our assumption is wrong and it must be the case that $\chi(G) \geq 3$. $\square$