How fast can we sort?

All the sorting algorithms we have seen so far are *comparison sorts*: only use comparisons to determine the relative order of elements.

- E.g., insertion sort, merge sort, quicksort, heapsort.

The best worst-case running time that we’ve seen for comparison sorting is $O(n \log n)$.

*Is $O(n \log n)$ the best we can do?*

*Decision trees* can help us answer this question.

---

**Decision-tree model**

A decision tree models the execution of any comparison sorting algorithm:

- One tree per input size $n$.
- The tree contains *all* possible comparisons (= if-branches) that could be executed for any input of size $n$.
- The tree contains *all* comparisons along all possible instruction traces (= control flows) for all inputs of size $n$.
- For one input, only one path to a leaf is executed.
- Running time = length of the path taken.
- Worst-case running time = height of tree.

---

**Decision-tree for insertion sort**

Sort $\langle a_1, a_2, a_3 \rangle$

Each internal node is labeled $a_i; a_j$ for $i, j \in \{1, 2, \ldots, n\}$.

- The left subtree shows subsequent comparisons if $a_i < a_j$.
- The right subtree shows subsequent comparisons if $a_i \geq a_j$.
Each internal node is labeled \( a_i : a_j \) for \( i, j \in \{1, 2, \ldots, n\} \).

- The left subtree shows subsequent comparisons if \( a_i < a_j \).
- The right subtree shows subsequent comparisons if \( a_i \geq a_j \).
**Decision-tree model**

*A decision tree models the execution of any comparison sorting algorithm:*

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- Running time = length of the path taken.
- Worst-case running time = height of tree.

**Lower bound for comparison sorting**

**Theorem.** Any decision tree that can sort $n$ elements must have height $\Omega(n \log n)$.

**Proof.** The tree must contain $\geq n!$ leaves, since there are $n!$ possible permutations. A height-$h$ binary tree has $\leq 2^h$ leaves. Thus, $n! \leq 2^h$.

\[ n! \geq \log(n!) \geq \log((n/2)^{n/2}) = n/2 \log n/2 \]

\[ \Rightarrow h \in \Omega(n \log n) \]
**Lower bound for comparison sorting**

**Corollary.** Heapsort and merge sort are asymptotically optimal comparison sorting algorithms.

**Sorting in linear time**

**Counting sort:** No comparisons between elements.

- **Input:** $A[1..n]$, where $A[j] \in \{1, 2, \ldots, k\}$.
- **Output:** $B[1..n]$, sorted.
- **Auxiliary storage:** $C[1..k]$.

**Counting sort**

1. for $i \leftarrow 1$ to $k$
   do $C[i] \leftarrow 0$
2. for $j \leftarrow 1$ to $n$
   do $C[A[j]] \leftarrow C[A[j]] + 1$  \( \triangleright \ C[i] = |\{\text{key} = i\}| \)
3. for $i \leftarrow 2$ to $k$
   do $C[i] \leftarrow C[i] + C[i-1]$  \( \triangleright \ C[i] = |\{\text{key} \leq i\}| \)
4. for $j \leftarrow n$ downto 1
   do $B[C[A[j]]] \leftarrow A[j]$
   $C[A[j]] \leftarrow C[A[j]] - 1$

**Counting-sort example**

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
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<tbody>
<tr>
<td>4 1 3 4 3</td>
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</table>

- $A$: 4 1 3 4 3
- $B$: 
- $C$:
Loop 1

1. for $i \leftarrow 1$ to $k$
   do $C[i] \leftarrow 0$

Loop 2

2. for $j \leftarrow 1$ to $n$
   do $C[A[j]] \leftarrow C[A[j]] + 1$ ▷ $C[i] = |\{key = i\}|$

2/24/09 CS 3343 Analysis of Algorithms
Loop 2

A: 4 1 3 4 3
B: 

2. for $j \leftarrow 1$ to $n$
   do $C[A[j]] \leftarrow C[A[j]] + 1$ \quad $\triangleright C[i] = |\{key = i\}|$

3. for $i \leftarrow 2$ to $k$
   do $C[i] \leftarrow C[i] + C[i-1]$ \quad $\triangleright C[i] = |\{key \leq i\}|$

Loop 3

A: 4 1 3 4 3
B: 

3. for $i \leftarrow 2$ to $k$
   do $C[i] \leftarrow C[i] + C[i-1]$ \quad $\triangleright C[i] = |\{key \leq i\}|$
Loop 3

3. for $i \leftarrow 2$ to $k$
   do $C[i] \leftarrow C[i] + C[i-1]$ \quad \triangleright \quad C[i] = |\{\text{key} \leq i\}|

Loop 4

4. for $j \leftarrow n$ downto 1
   do $B[C[A[j]]] \leftarrow A[j]$
     $C[A[j]] \leftarrow C[A[j]] - 1$
for $j \leftarrow n$ downto 1
do $B[C[A[j]]] \leftarrow A[j]$
$C[A[j]] \leftarrow C[A[j]] - 1$
for $j \leftarrow n \text{ downto } 1$
  do $B[C[A[j]]] \leftarrow A[j]$
  $C[A[j]] \leftarrow C[A[j]] - 1$

4.

Analysis

$\Theta(k)$
  1. for $i \leftarrow 1 \text{ to } k$
     do $C[i] \leftarrow 0$
  2. for $j \leftarrow 1 \text{ to } n$
     do $C[A[j]] \leftarrow C[A[j]] + 1$
  3. for $i \leftarrow 2 \text{ to } k$
     do $C[i] \leftarrow C[i] + C[i-1]$
  4. for $j \leftarrow n \text{ downto } 1$
     do $B[C[A[j]]] \leftarrow A[j]$
     $C[A[j]] \leftarrow C[A[j]] - 1$

$\Theta(n + k)$
Running time

If $k = O(n)$, then counting sort takes $\Theta(n)$ time.
• But, sorting takes $\Omega(n \log n)$ time!
• Where’s the fallacy?

Answer:
• Comparison sorting takes $\Omega(n \log n)$ time.
• Counting sort is not a comparison sort.
• In fact, not a single comparison between elements occurs!

Stable sorting

Counting sort is a stable sort: it preserves the input order among equal elements.

Exercise: What other sorts have this property?

Radix sort

• Origin: Herman Hollerith’s card-sorting machine for the 1890 U.S. Census. (See Appendix 4.)
• Digit-by-digit sort.
• Hollerith’s original (bad) idea: sort on most-significant digit first.
• Good idea: Sort on least-significant digit first with an auxiliary stable sorting algorithm (like counting sort).
Correctness of radix sort

Induction on digit position

- Assume that the numbers are sorted by their low-order $t - 1$ digits.
- Sort on digit $t$

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Two numbers that differ in digit $t$ are correctly sorted.

Two numbers equal in digit $t$ are put in the same order as the input ⇒ correct order.

Correctness of radix sort

Induction on digit position

- Assume that the numbers are sorted by their low-order $t - 1$ digits.
- Sort on digit $t$

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Two numbers that differ in digit $t$ are correctly sorted.

Two numbers equal in digit $t$ are put in the same order as the input ⇒ correct order.

Analysis of radix sort

- Sort $n$ computer words of $b$ bits each.
- View each word as having $b/r$ base-$2^r$ digits.

Example: 32-bit word ($b=32$)

$r = 1$: 32 base-2 digits ⇒ $b/r = 32$ passes of counting sort on base-2 digits

$r = 8$: 32/8 base-2^8 digits ⇒ $b/r = 4$ passes of counting sort on base-2^8 digits

$r = 16$: 32/16 base-2^{16}$ digits ⇒ $b/r = 2$ passes of counting sort on base-2^{16}$ digits
Analysis of radix sort

- Sort $n$ computer words of $b$ bits each.
- View each word as having $b/r$ base-$2^r$ digits.
- Assume counting sort is the auxiliary stable sort.
- Make $b/r$ passes of counting sort on base-$2^r$ digits

How many passes should we make?

Analysis (continued)

Recall: Counting sort takes $\Theta(n + k)$ time to sort $n$ numbers in the range from 0 to $k – 1$.

- If each $b$-bit word is broken into $r$-bit pieces, each pass of counting sort takes $\Theta(n + 2^r)$ time.
- Since there are $b/r$ passes, we have

$$T(n, b) = \Theta\left(\frac{b}{r} \left(n + 2^r\right)\right).$$

- Choose $r$ to minimize $T(n, b)$:
  Increasing $r$ means fewer passes, but as $r >> \log n$, the time grows exponentially.

Choosing $r$

$$T(n, b) = \Theta\left(\frac{b}{r} \left(n + 2^r\right)\right)$$

Minimize $T(n, b)$ by differentiating and setting to 0.

Or, just observe that we don’t want $2^r \gg n$, and there’s no harm asymptotically in choosing $r$ as large as possible subject to this constraint.

Choosing $r = \log n$ implies $T(n, b) = \Theta(bn/\log n)$.

Radix Sort with optimized $r$

- Assume counting sort is the auxiliary stable sort.
- Sort $n$ computer words of $b$ bits each.

The runtime of radix sort is: $T(n, b) = \Theta(bn/\log n)$.

- Example:
  For numbers in the range from 0 to $n^d – 1$, we have $b = d \log n \Rightarrow$ radix sort runs in $\Theta(dn)$ time.
- Notice that counting sort runs in $O(n+k)$ time, where all numbers are in the range 1 through $k$. 
Conclusions

In practice, radix sort is fast for large inputs, as well as simple to code and maintain.

**Example** (32-bit numbers):
- At most 3 passes when sorting \( \geq 2000 \) numbers.
- Merge sort and quicksort do at least \( \lceil \log 2000 \rceil \) = 11 passes.

**Downside:** Unlike quicksort, radix sort displays little locality of reference, and thus a well-tuned quicksort fares better on modern processors, which feature steep memory hierarchies.

Appendix: Punched-card technology

- Herman Hollerith (1860-1929)
- Punched cards
- Hollerith’s tabulating system
- Operation of the sorter
- Origin of radix sort
- “Modern” IBM card

Herman Hollerith (1860-1929)

- The 1880 U.S. Census took almost 10 years to process.
- While a lecturer at MIT, Hollerith prototyped punched-card technology.
- His machines, including a “card sorter,” allowed the 1890 census total to be reported in 6 weeks.
- He founded the Tabulating Machine Company in 1911, which merged with other companies in 1924 to form International Business Machines.

Punched cards

- Punched card = data record.
- Hole = value.
- Algorithm = machine + human operator.
Hollerith’s tabulating system
- Pantograph card punch
- Hand-press reader
- Dial counters
- Sorting box

Figure from [Howells 2000].

Operation of the sorter
- An operator inserts a card into the press.
- Pins on the press reach through the punched holes to make electrical contact with mercury-filled cups beneath the card.
- Whenever a particular digit value is punched, the lid of the corresponding sorting bin lifts.
- The operator deposits the card into the bin and closes the lid.
- When all cards have been processed, the front panel is opened, and the cards are collected in order, yielding one pass of a stable sort.

Origin of radix sort
Hollerith’s original 1889 patent alludes to a most-significant-digit-first radix sort:

“The most complicated combinations can readily be counted with comparatively few counters or relays by first assorting the cards according to the first items entering into the combinations, then reasorting each group according to the second item entering into the combination, and so on, and finally counting on a few counters the last item of the combination for each group of cards.”

Least-significant-digit-first radix sort seems to be a folk invention originated by machine operators.

“Modern” IBM card
- One character per column.

So, that’s why text windows have 80 columns!