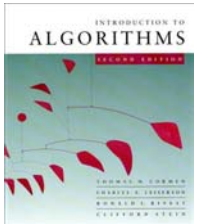
CS 2233 -- Fall 2009



Graphs

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Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk

Graphs

Definition.

A (simple) ***undirected graph*** $G = (V, E)$ consists of

- a set V of ***vertices*** (singular: ***vertex***), and
- a set E of ***edges*** consisting of 2-element subsets of V.

(unordered pairs)

A ***directed graph*** **(*digraph*)** $G = (V, E)$ consists of

- a set V of ***vertices***, and
- a set $E \subseteq V \times V$ of ***edges***.

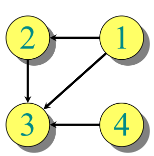
(ordered pairs)

In either case, we have $|E| \in O(|V|^2)$.

Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,..\, n, 1 \,..\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$

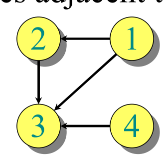


A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.

Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = \deg(v)$. (degree)

For digraphs, $|Adj[v]| = \deg^+(v)$. (out-degree)

Adjacency-list representation

Handshaking Lemma:

Every edge is counted twice

- For undirected graphs:

  $\sum_{v\in V} \deg(v) = 2|E|$

- For digraphs:

  $\sum_{v\in V} \deg^-(v) + \sum_{v\in V} \deg^+(v) = 2\,|\,E\,|$

  in-degree   out-degree

$\Rightarrow$ adjacency lists use $\Theta(|V| + |E|)$ storage

$\Rightarrow$ a ***sparse*** representation

$\Rightarrow$ We usually use this representation, unless stated otherwise

Paths, Cycles, Connectivity

Let $G=(V,E)$ be a directed (or undirected) graph

- A **path** of length k in G is a sequence of vertices $v_1, v_2,\ldots,v_{k+1}$ such that $(v_i,v_{\{i+1\}})\in E$ (or $\{v_i,v_{\{i+1\}}\} \in E$ if G is undirected) for all $i\in\{1,\ldots,k\}$.
- A path is **simple** if all vertices in the path are distinct.
- A path $v_1, v_2,\ldots,v_{k+1}$ forms a **cycle** if $v_1=v_{k+1}$ and $k\geq 2$.
- An undirected graph is **connected** if for every pair $u,v\in V$ there is a path from u to v. (Note that then there is also a path from v to u.)
- A directed graph is **strongly connected** if for every pair $u,v\in V$ there is a path from u to v and there is a path from v to u.
- The **(strongly) connected components** of a graph are the equivalence classes of vertices under this reachability relation.
- A graph with no cycles is **acyclic**.
  - An undirected connected acyclic graph is called a **tree**.
  - An undirected acyclic graph is a **forest** (= set of trees).
  - A **rooted tree** is a tree that has one distinguished **root** vertex.