CMPS 2200 – Fall 2017

P and NP (Millenium problem)

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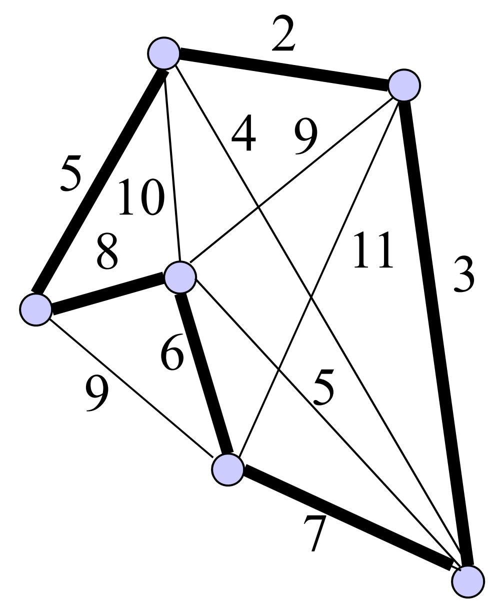
Slides courtesy of Piotr Indyk with additions by Carola Wenk

We have seen so far

- Algorithms for various problems
  - Running times $O(nm^2)$, $O(n^2)$, $O(n \log n)$, $O(n)$, etc.
  - I.e., polynomial in the input size
- Can we solve all (or most of) interesting problems in polynomial time ?
- Not really…

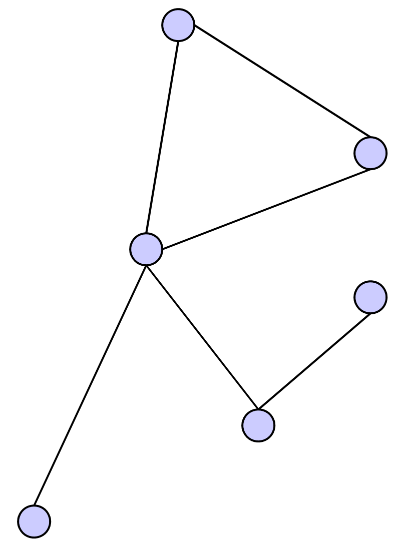
Example difficult problem

- Traveling Salesperson Problem (TSP; optimization variant)
  - **Input:** Undirected graph with weights on edges
  - **Output:** Shortest tour that visits each vertex exactly once
- Best known algorithm: $O(n\,2^n)$ time.



Another difficult problem

- Clique (optimization variant):
  - **Input:** Undirected graph $G=(V,E)$
  - **Output:** Largest subset C of V such that every pair of vertices in C has an edge between them (C is called a *clique*)
- Best known algorithm: $O(n\ 2^n)$ time



What can we do ?

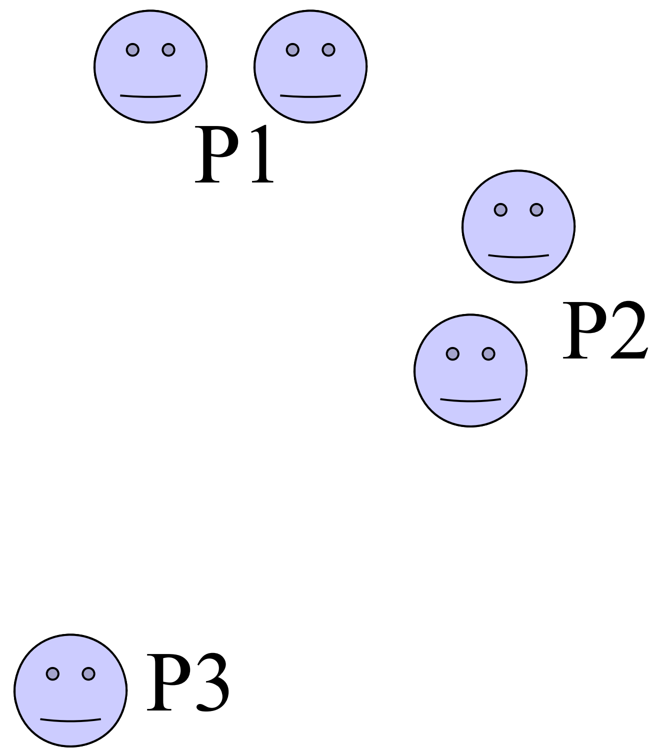
- Spend more time designing algorithms for those problems
  - People tried for a few decades, no luck
- Prove there is no polynomial time algorithm for those problems
  - Would be great
  - Seems *really* difficult
  - Best lower bounds for "natural" problems:
    - $\Omega(n^2)$ for restricted computational models
    - $4.5n$ for unrestricted computational models

What else can we do ?

- Show that those hard problems are essentially equivalent. I.e., if we can solve one of them in polynomial time, then all others can be solved in polynomial time as well.
- Works for at least 10 000 hard problems

The benefits of equivalence

- Combines research efforts
- If one problem has a polynomial time solution, then all of them do
- More realistically: Once an exponential lower bound is shown for one problem, it holds for all of them



Summing up

- If we show that a problem $\prod$ is equivalent to ten thousand other well studied problems without efficient algorithms, then we get a very strong evidence that $\prod$ is hard.
- We need to:
  1. Identify the class of problems of interest
     $\rightarrow$ Decision problems, NP
  2. Define the notion of equivalence
     $\rightarrow$ Polynomial-time reductions
  3. Prove the equivalence(s)

Decision Problem

- Decision problems: answer YES or NO.
- Example: **Search Problem** Π_{Search}
  Given an unsorted set S of n numbers and a number key, is key contained in S?
- Input is x=(S,key)
- Possible algorithms that solve $\Pi_{\text{Search}}(x)$:
  - $A_1(x)$: Linear search algorithm. $O(n)$ time
  - $A_2(x)$: Sort the array and then perform binary search. $O(n \log n)$ time
  - $A_3(x)$: Compute all possible subsets of S (2^n many) and check each subset if it contains key. $O(n2^n)$ time.

Decision problem vs. optimization problem

3 variants of Clique:

1. **Input:** Undirected graph $G=(V,E)$, and an integer $k \geq 0$.
   Output: Does G contain a clique C such that $|C| \geq k$?
2. **Input:** Undirected graph $G=(V,E)$
   Output: Largest integer k such that G contains a clique C with $|C|=k$.
3. **Input:** Undirected graph $G=(V,E)$
   Output: Largest clique C of V.

3. is harder than **2.** is harder than **1.** So, if we reason about the decision problem (**1.**), and can show that it is hard, then the others are hard as well. Also, every algorithm for **3.** can solve **2.** and **1.** as well.

Decision problem vs. optimization problem (cont.)

Theorem:

a) If **1.** can be solved in polynomial time, then **2.** can be solved in polynomial time.

b) If **2.** can be solved in polynomial time, then **3.** can be solved in polynomial time.

Proof:

a) Run **1.** for values $k = 1 \ldots n$. Instead of linear search one could also do binary search.

b) Run **2.** to find the size k_{opt} of a largest clique in G. Now check one edge after the other. Remove one edge from G, compute the new size of the largest clique in this new graph. If it is still k_{opt} then this edge is not necessary for a clique. If it is less than k_{opt} then it is part of the clique.

Class of problems: NP

- Decision problems: answer YES or NO. E.g.,"is there a tour of length $\leq K$" ?
- Solvable in *non-deterministic polynomial* time:
  - Intuitively: the solution can be verified in polynomial time
  - E.g., if someone gives us a tour T, we can verify in *polynomial* time if T is a tour of length $\leq K$.
- Therefore, the decision variant of TSP is in NP.

Formal definitions of P and NP

- A decision problem $\prod$ is solvable in polynomial time (or $\prod \in \mathrm{P}$), if there is a polynomial time algorithm $A(.)$ such that for any input x:

$$\prod(x)=\text{YES} \text{ iff } A(x)=\text{YES}$$

- A decision problem $\prod$ is solvable in non-deterministic polynomial time (or $\prod \in \mathrm{NP}$), if there is a polynomial time algorithm $A(. \, , .)$ such that for any input x:

$$\prod(x)=\text{YES} \text{ iff there exists a certificate } y \text{ of size } \text{poly}(|x|) \text{ such that } A(x,y)=\text{YES}$$

Examples of problems in NP

- Is "Does there exist a clique in G of size $\geq K$" in NP ?

  Yes: $A(x,y)$ interprets x as (G,K), y as a set C, and checks if all vertices in C are adjacent and if $|C|\geq K$
- Is Sorting in NP ?

  No, not a decision problem.
- Is "Sortedness" in NP ?

  Yes: ignore y, and check if the input x is sorted.

Summing up

- If we show that a problem $\prod$ is equivalent to ten thousand other well studied problems without efficient algorithms, then we get a very strong evidence that $\prod$ is hard.
- We need to:
  1. Identify the class of problems of interest
     $\rightarrow$ Decision problems, NP
  2. Define the notion of equivalence
     $\rightarrow$ Polynomial-time reductions
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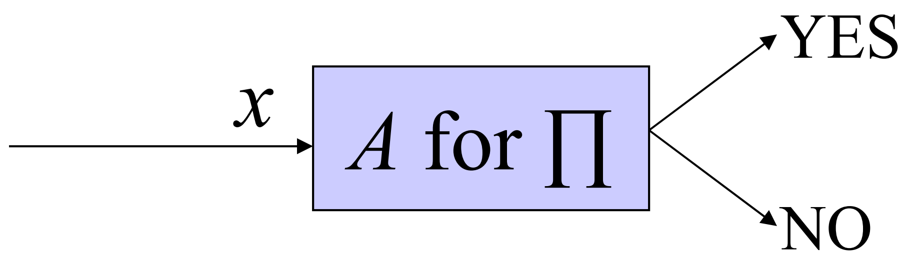
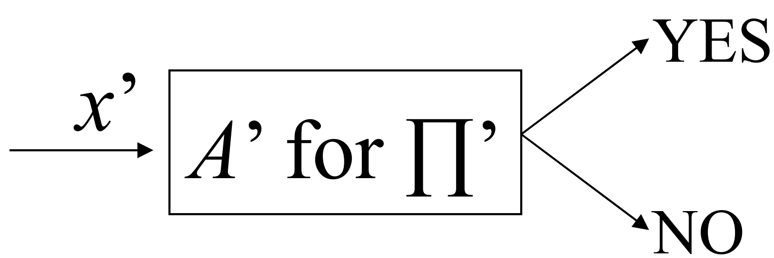
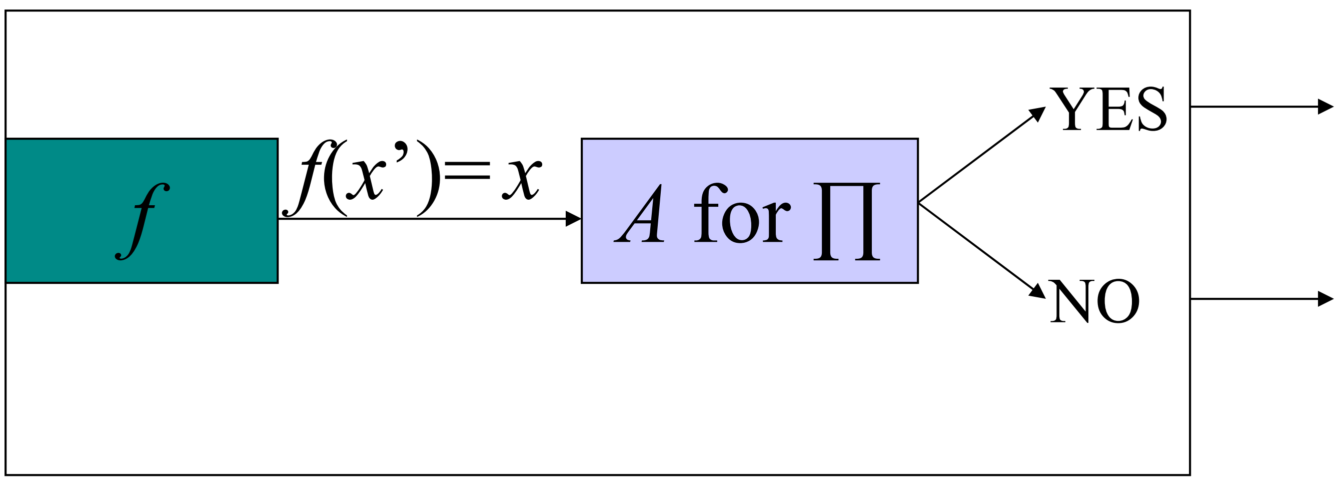
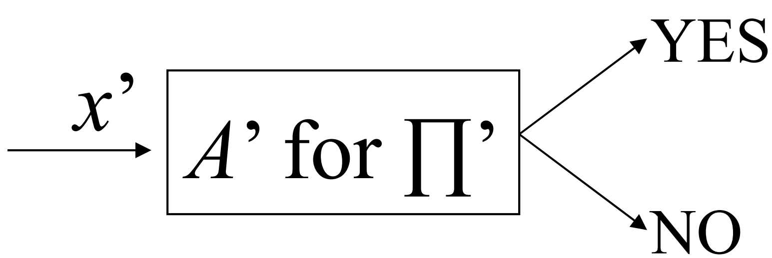
$\prod' \leq \prod$: Reduce $\prod'$ to $\prod$



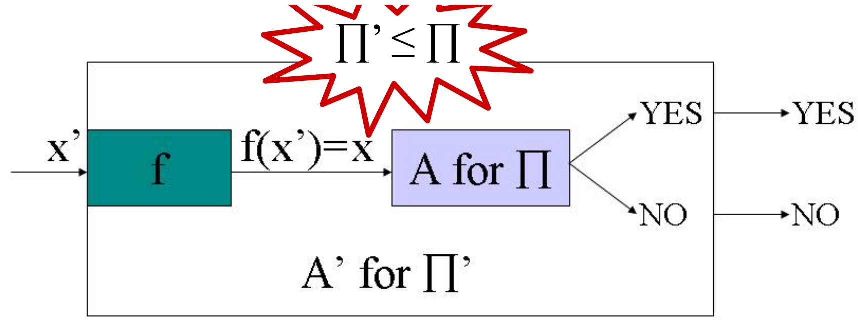


$\prod$' ≤ $\prod$: Reduce $\prod$' to $\prod$





Reductions



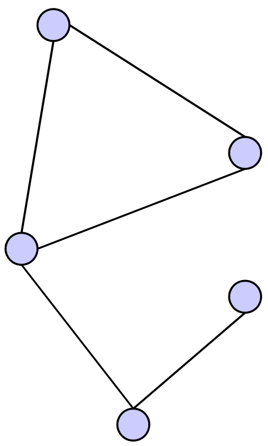
- ∏' is *polynomial time reducible* to ∏ (∏' ≤ ∏) iff
  1. there is a polynomial time function f that maps inputs *x'* for ∏' into inputs *x* for ∏,
  2. such that for any *x'*:

$$\prod'(x') = \prod(f(x'))$$

(or in other words ∏'(*x'*)=YES iff ∏(*f*(*x'*)=YES)

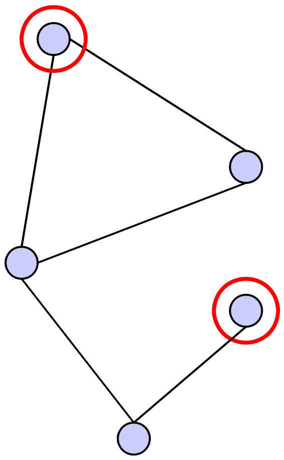
Clique again

- Clique (decision variant):
  - **Input:** Undirected graph $G=(V,E)$, and an integer $K\geq0$
  - **Output:** Is there a clique C, i.e., a subset C of V such that **every** pair of vertices in C has an edge between them, such that $|C|\geq K$?

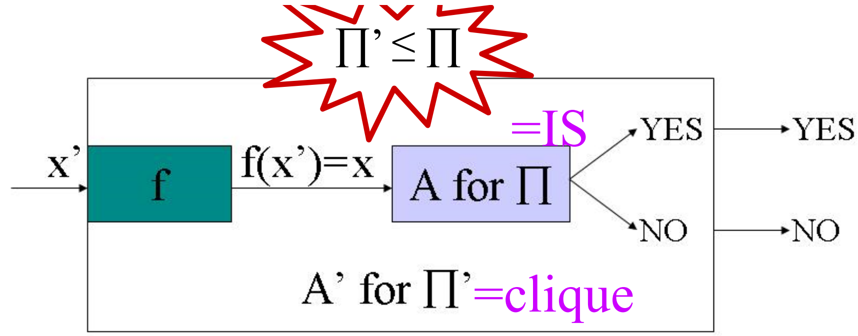


Independent Set (IS)

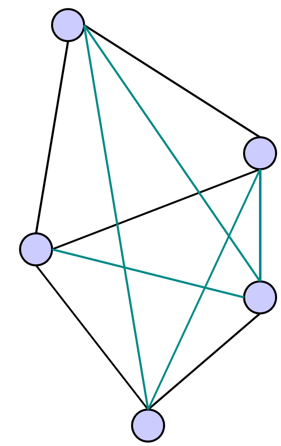
- Independent Set (decision variant):
  - **Input:** Undirected graph $G=(V,E)$, and an integer $K\geq 0$
  - **Output:** Is there a subset S of V, $|S|\geq K$ such that **no** pair of vertices in S has an edge between them? (S is called an *independent set*)



$\underbrace{\text{Clique}}_{\Pi'} \le \underbrace{\text{IS}}_{\Pi}$



- Given an input $\overbrace{G=(V,E), K}^{x'}$ to Clique, need to construct an input $\underbrace{G'=(V',E'), K'}_{f(x')=x}$ to IS, such that G has clique of size $\geq K$ iff G' has IS of size $\geq K'$.
- Construction: $K'=K, V'=V, E'=\overline{E}$
- Reason: C is a clique in G iff it is an IS in G's complement.



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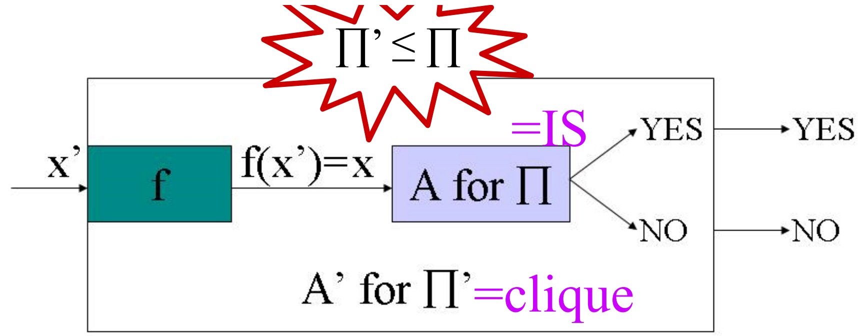

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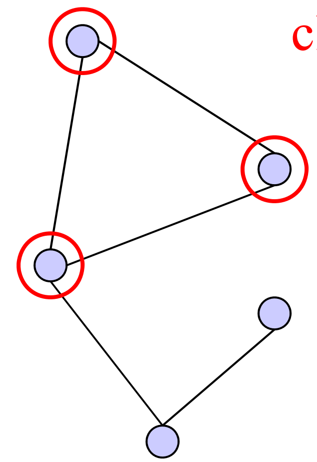
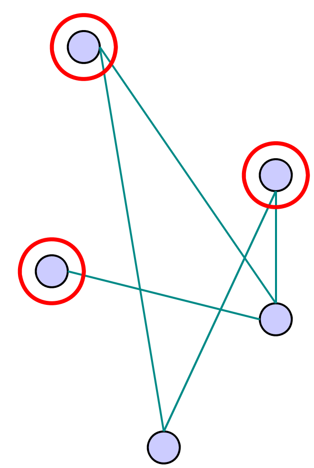
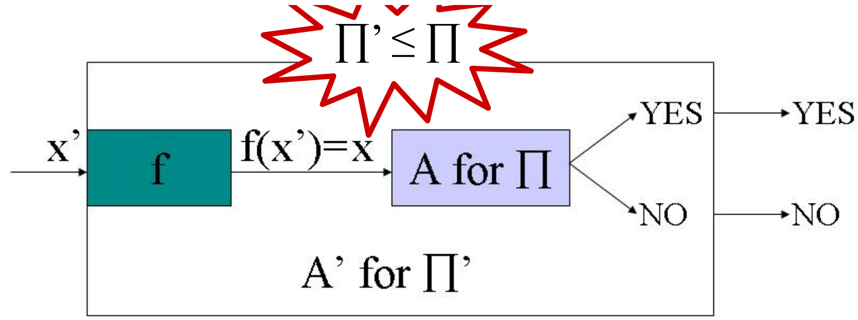
clique



IS



Reductions



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  1. there is a polynomial time function f that maps inputs x' for ∏' into inputs x for ∏,
  2. such that for any x':

$$\prod'(x') = \prod(f(x'))$$

(or in other words ∏'(x')=YES iff ∏(f(x')=YES)

- Fact 1: if ∏∈P and ∏' ≤ ∏ then ∏'∈P
- Fact 2: if ∏∈NP and ∏' ≤ ∏ then ∏'∈NP
- Fact 3 (transitivity):

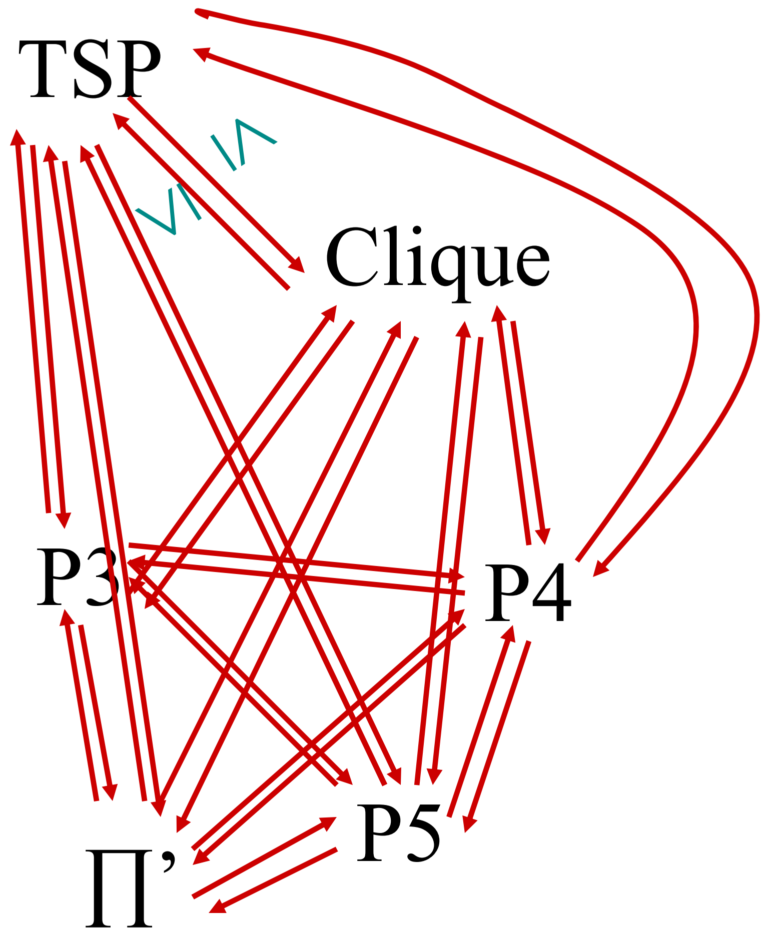
  if ∏'' ≤ ∏' and ∏' ≤ ∏ then ∏'' ≤ ∏

Recap

- We defined a large class of interesting problems, namely NP
- We have a way of saying that one problem is not harder than another ($\Pi' \leq \Pi$)
- Our goal: show equivalence between hard problems

Showing equivalence between difficult problems

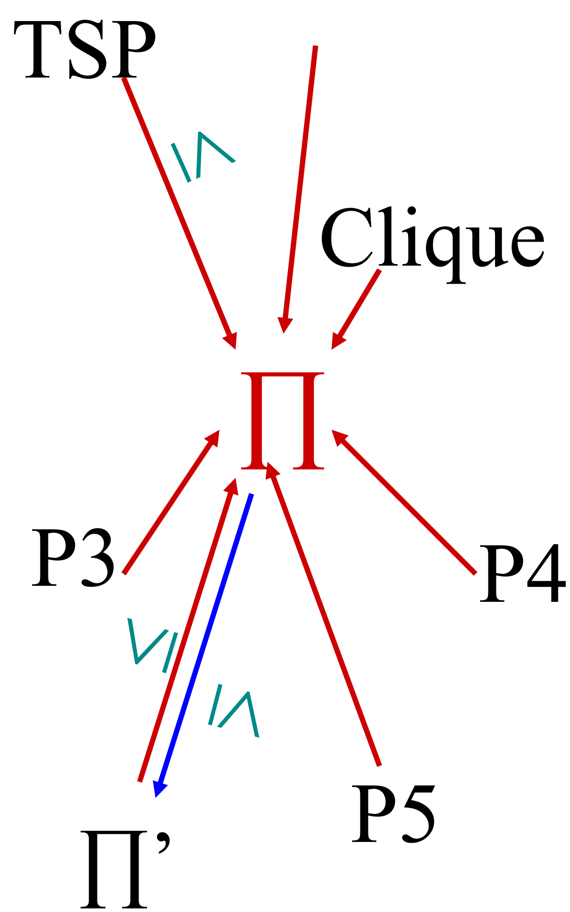
- Options:
  - Show reductions between all pairs of problems
  - Reduce the number of reductions using transitivity of "$\leq$"



Showing equivalence between difficult problems

- Options:
  - Show reductions between all pairs of problems
  - Reduce the number of reductions using transitivity of "$\leq$"
  - Show that *all* problems in NP are reducible to a *fixed* $\prod$.

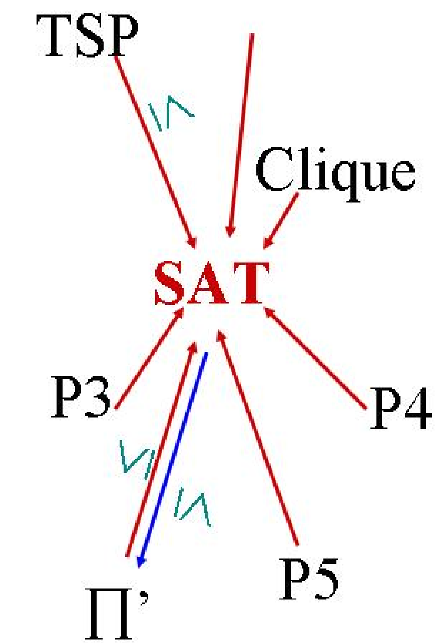
  To show that some problem $\prod' \in NP$ is equivalent to all difficult problems, we only show $\prod \leq \prod'$.



The first problem $\prod$

- Satisfiability problem (SAT):
  - Given: a formula φ with m clauses over n variables, e.g., $x_1 \vee x_2 \vee x_5 \,,\, x_3 \vee \neg x_5$
  - Check if there exists TRUE/FALSE assignments to the variables that makes the formula satisfiable

SAT is NP-complete



- Fact: SAT $\in$NP
- Theorem [Cook'71]: For any $\prod' \in$NP we have $\prod' \leq$ SAT.
- Definition: A problem $\prod$ such that for any $\prod' \in$NP we have $\prod' \leq \prod$, is called *NP-hard*
- Definition: An NP-hard problem that belongs to NP is called *NP-complete*
- Corollary: SAT is NP-complete.

Plan of attack:

- Show that the problems below are in NP, and show a sequence of reductions:

SAT

↓

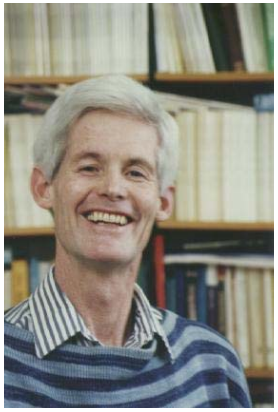
Clique

↓

Independent set

↓

Vertex cover



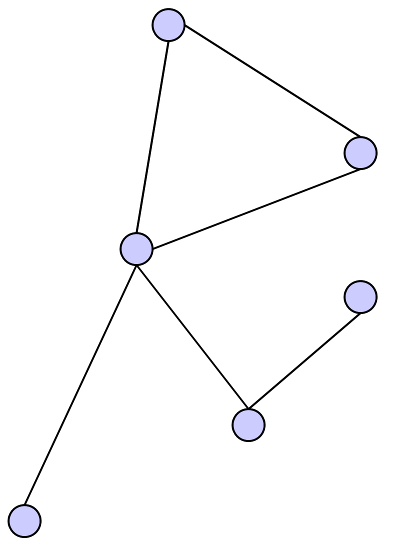
(thanks, Steve ☺)

Follow from Cook's Theorem

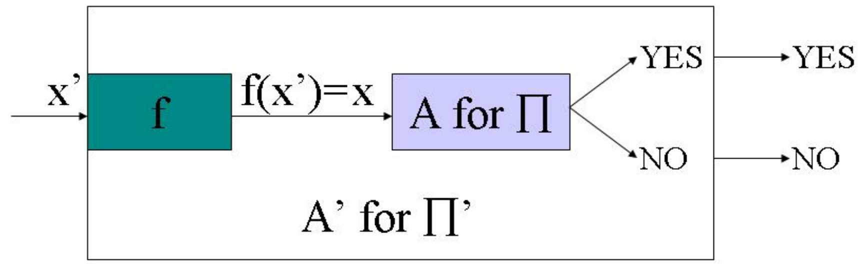
- Conclusion: all of the above problems are NP-complete

Clique again

- Clique (decision variant):
  - **Input:** Undirected graph $G=(V,E)$, and an integer $K\geq 0$
  - **Output:** Is there a clique C, i.e., a subset C of V such that every pair of vertices in C has an edge between them, such that $|C|\geq K$?



SAT ≤ Clique



- Given a $\overbrace{\text{SAT formula } \varphi=C_1,\ldots,C_m}^{x'}$ over $x_1,\ldots,x_n$, we need to produce $\underbrace{G=(V,E) \text{ and } K,}_{f(x')=x}$

  such that φ satisfiable iff G has a clique of size $\geq K$.
- Notation: a literal is either x_i or $\neg x_i$

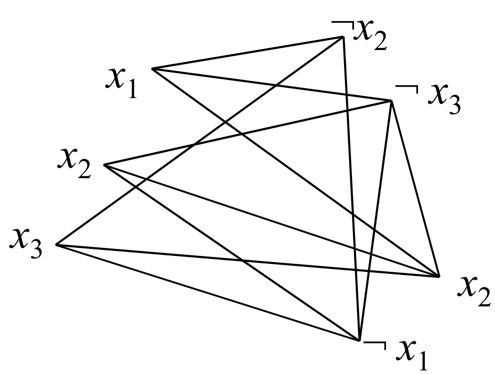
SAT ≤ Clique reduction

- For each literal t occurring in φ, create a vertex v_{t}
- Create an edge $v_t - v_{t'}$ iff:
  - t and t' are not in the same clause, and
  - t is not the negation of t'

SAT ≤ Clique example

Edge $v_t - v_{t'} \Leftrightarrow$	• t and t' are not in the same clause, and • t is not the negation of t'

- Formula: $x_1 \vee x_2 \vee x_3\,,\ \neg x_2 \vee \neg x_3,\ \neg x_1 \vee x_2$
- Graph:



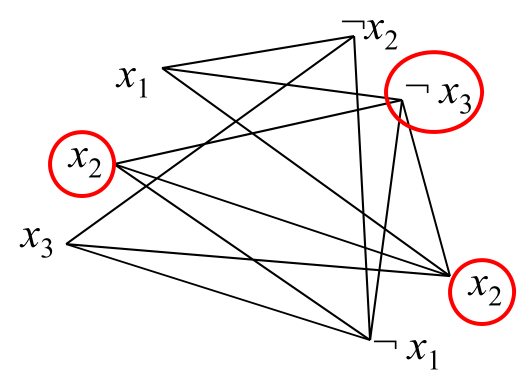
- Claim: φ satisfiable iff G has a clique of size $\geq m$

Proof

Edge $v_t - v_{t'} \Leftrightarrow$	• t and t' are not in the same clause, and • t is not the negation of t'

- "→" part of Claim:
  - Take any assignment that satisfies φ.

    E.g., x_1=F, x_2=T, x_3=F
  - Let the set C contain one satisfied literal per clause
  - C is a clique

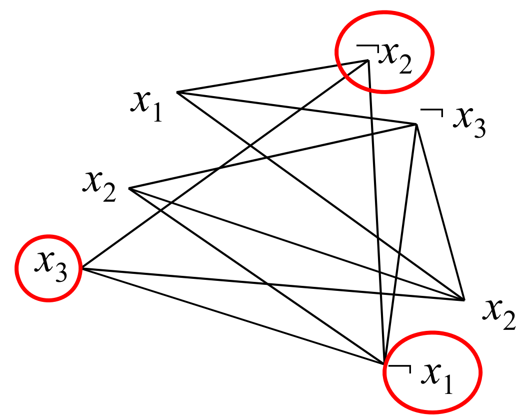


Proof

Edge $v_t - v_{t'} \Leftrightarrow$	• t and t' are not in the same clause, and • t is not the negation of t'

- "←" part of Claim:
  - Take any clique C of size $\geq m$ (i.e., $= m$)
  - Create a set of equations that satisfies selected literals.

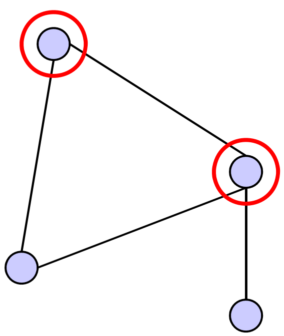
    E.g., x_3=T, x_2=F, x_1=F
  - The set of equations is consistent and the solution satisfies φ



Altogether

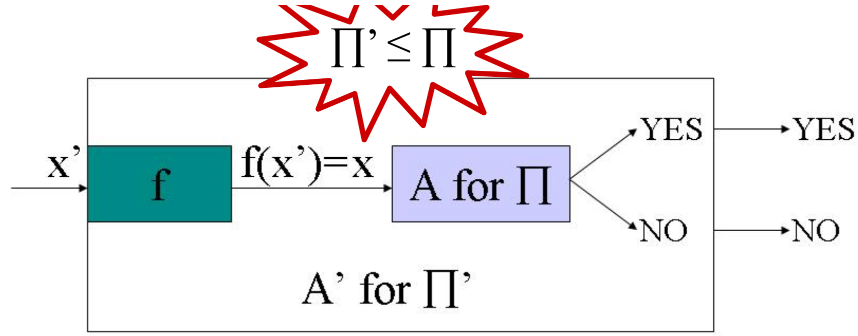
- We constructed a reduction that maps:
  - YES inputs to SAT to YES inputs to Clique
  - NO inputs to SAT to NO inputs to Clique
- The reduction works in polynomial time
- Therefore, SAT $\leq$ Clique $\rightarrow$ Clique NP-hard
- Clique is in NP $\rightarrow$ Clique is NP-complete

Vertex cover (VC)



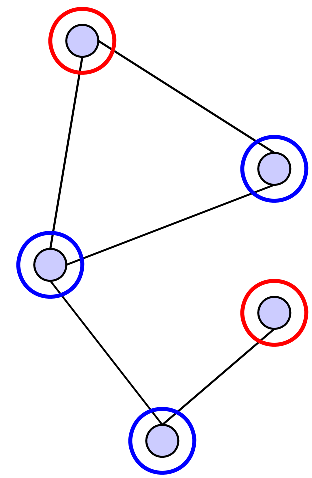
- Input: undirected graph $G=(V,E)$, and $K \geq 0$
- Output: is there a subset C of V, $|C| \leq K$, such that each edge in E is incident to at least one vertex in C.

$\underbrace{\text{IS}}_{\Pi'} \leq \underbrace{\text{VC}}_{\Pi}$



- Given an input $\overbrace{G=(V,E),\ K}^{x'}$ to IS, need to construct an input $\underbrace{G'=(V',E'),\ K'}_{f(x')=x}$ to VC, such that

  G has an IS of size $\geq K$ iff G' has VC of size $\leq K'$.



- Construction: $V'=V$, $E'=E$, $K'=|V|-K$
- Reason: S is an IS in G iff $V-S$ is a VC in G.