CMPS 2200 – Fall 2015

Divide-and-Conquer

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Slides courtesy of Charles Leiserson
with changes and additions by Carola Wenk

The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems of sizes that are fractions of the original problem size.
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.

Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3	5	7	8	9	12	15

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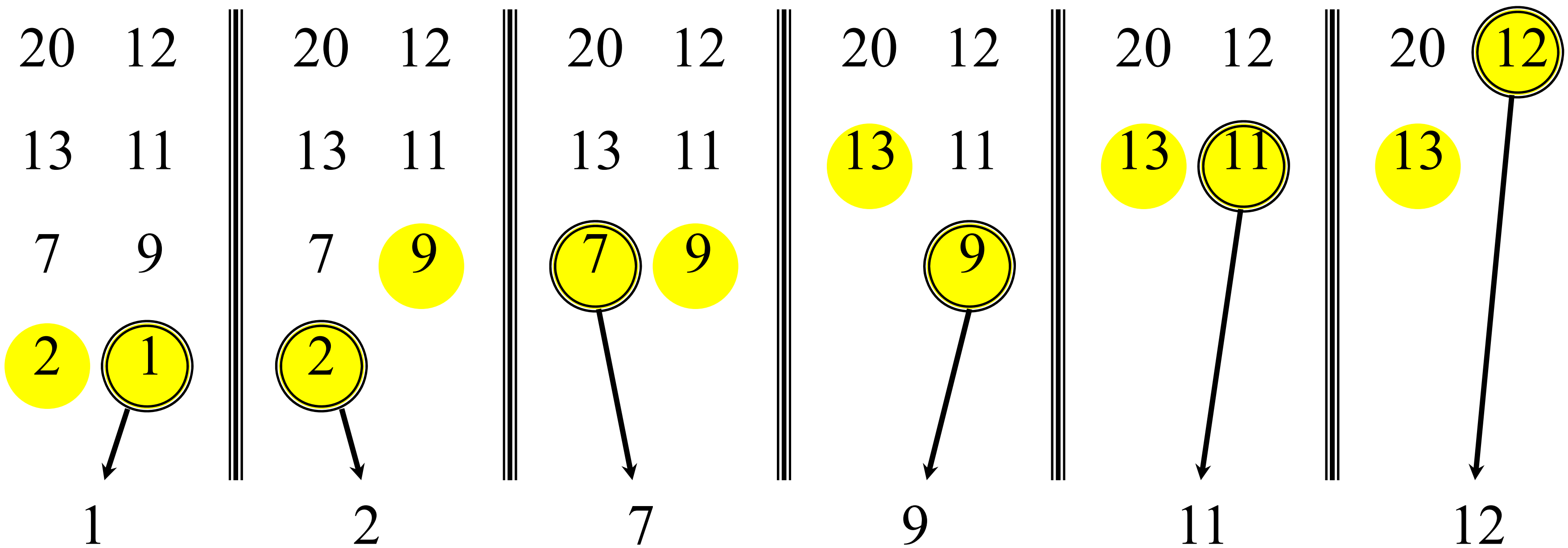
Merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays of size $n/2$
3. ***Combine:*** Linear-time key subroutine **MERGE**

MERGE-SORT ($A[0 \,..\, n\text{-}1]$)

1. If $n = 1$, done.
2. **MERGE-SORT** ($A[\,0 \,..\, \lceil n/2 \rceil \text{-}1]$)
3. **MERGE-SORT** ($A[\,\lceil n/2 \rceil \,..\, n\text{-}1\,]$)
4. "***Merge***" the 2 sorted lists.

Merging two sorted arrays



Time $dn \in \Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

$T(n)$	**MERGE-SORT** $(A[0 \,..\, n\text{-}1])$
d_0	1. If $n = 1$, done.
$T(n/2)$	2. **MERGE-SORT** $(A[\,0 \,..\, \lceil n/2 \rceil + 1\,])$
$T(n/2)$	3. **MERGE-SORT** $(A[\,\lceil n/2 \rceil \,..\, n\text{-}1\,])$
dn	4. "***Merge***" the 2 sorted lists.

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.

Recurrence for merge sort

$$T(n) = \begin{cases} d_0 \text{ if } n = 1; \\ 2T(n/2) + dn \text{ if } n > 1. \end{cases}$$

- But what does $T(n)$ solve to? I.e., is it $O(n)$ or $O(n^2)$ or $O(n^3)$ or …?

Recursion tree

Solve $T(n) = 2T(n/2) + dn$, where $d > 0$ is constant.

$$T(n)$$

Recursion tree

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$$dn$$

$$T(n/2) \qquad T(n/2)$$

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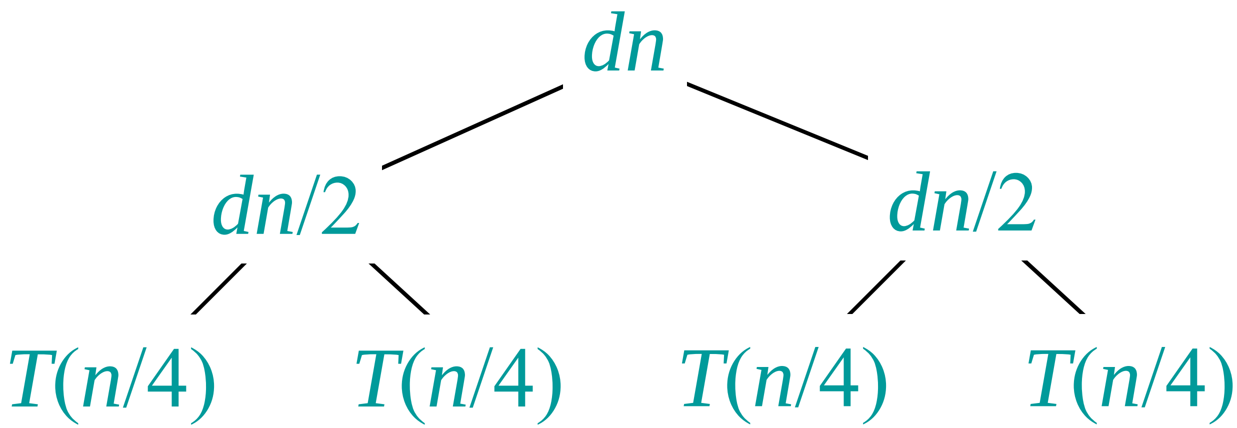

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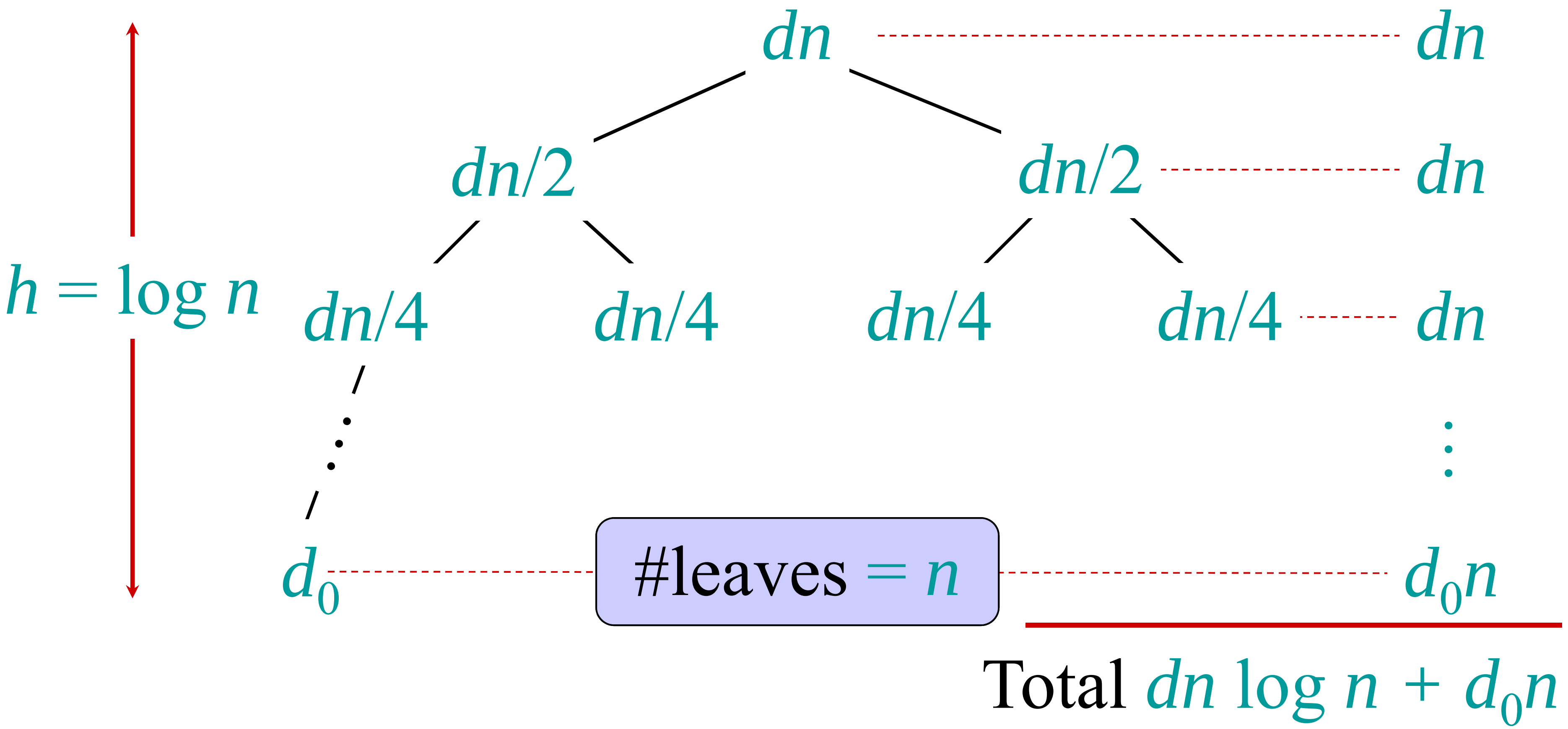

Mergesort Conclusions

- Merge sort runs in $\Theta(n \log n)$ time.
- $\Theta(n \log n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so. (Why not earlier?)

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- It is good for generating **guesses** of what the runtime could be.

But: Need to **verify** that the guess is correct.
→ Induction (substitution method)

Substitution method

The most general method to solve a recurrence (prove O and Ω separately):

1. ***Guess*** the form of the solution:
   (e.g. using recursion trees, or expansion)
2. ***Verify*** by induction (inductive step).
3. ***Solve*** for O-constants n_0 and c (base case of induction)