# CMPS 2200 – Fall 2015

# *Graphs*

**Carola Wenk**

Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk

# Graphs

**Definition.** A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set $V$ of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

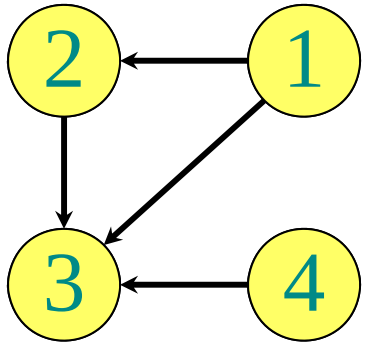
In an ***undirected graph*** $G = (V, E)$, the edge set $E$ consists of *unordered* pairs of vertices.

In either case, we have $|E| = O(|V|^2)$.
Moreover, if $G$ is connected, then $|E| \geq |V| - 1$.

# Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \ldots n, 1 \ldots n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$

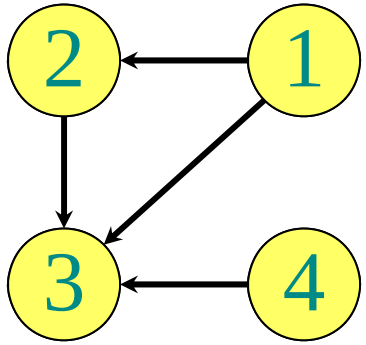



| $A$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.

# Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to $v$.




$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

# Adjacency-list representation

**Handshaking Lemma:**

*Every edge is counted twice*

- For undirected graphs:

  $\sum_{v \in V} degree(v) = 2|E|$

- For digraphs:

  $\sum_{v \in V} in\text{-}degree(v) + \sum_{v \in V} out\text{-}degree(v) = 2 \mid E \mid$

$\Rightarrow$ adjacency lists use $\Theta(|V| + |E|)$ storage

$\Rightarrow$ a ***sparse*** representation

$\Rightarrow$ We usually use this representation, unless stated otherwise

# Graph Traversal

Let $G=(V,E)$ be a (directed or undirected) graph, given in adjacency list representation.

$|V| = n$ , $|E| = m$

A graph traversal visits every vertex:

- Breadth-first search (BFS)
- Depth-first search (DFS)

# Breadth-First Search (BFS)

```
BFS(G=(V,E))
    Mark all vertices in G as "unvisited" // time=0
    Initialize empty queue Q
    for each vertex v ∈ V do
        if v is unvisited
            visit v // time++
            Q.enqueue(v)
            BFS_iter(G)
```

```
BFS_iter(G)
    while Q is non-empty do
        v = Q.dequeue()
        for each w adjacent to v do
            if w is unvisited
                visit w   // time++
                Add edge (v,w) to T
                Q.enqueue(w)
```

# Example of breadth-first search

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while Q is non-empty do
    v = Q.dequeue()
    for each w adjacent to v do
        if w is unvisited
            visit w    // time++
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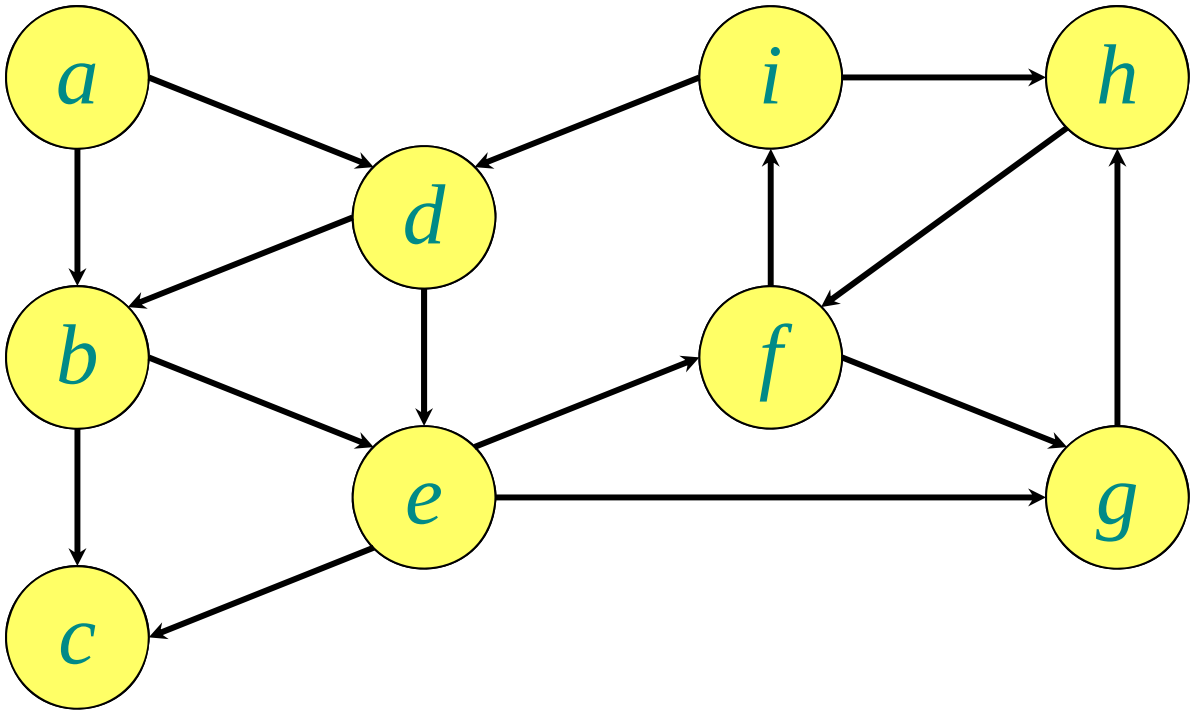


Q:

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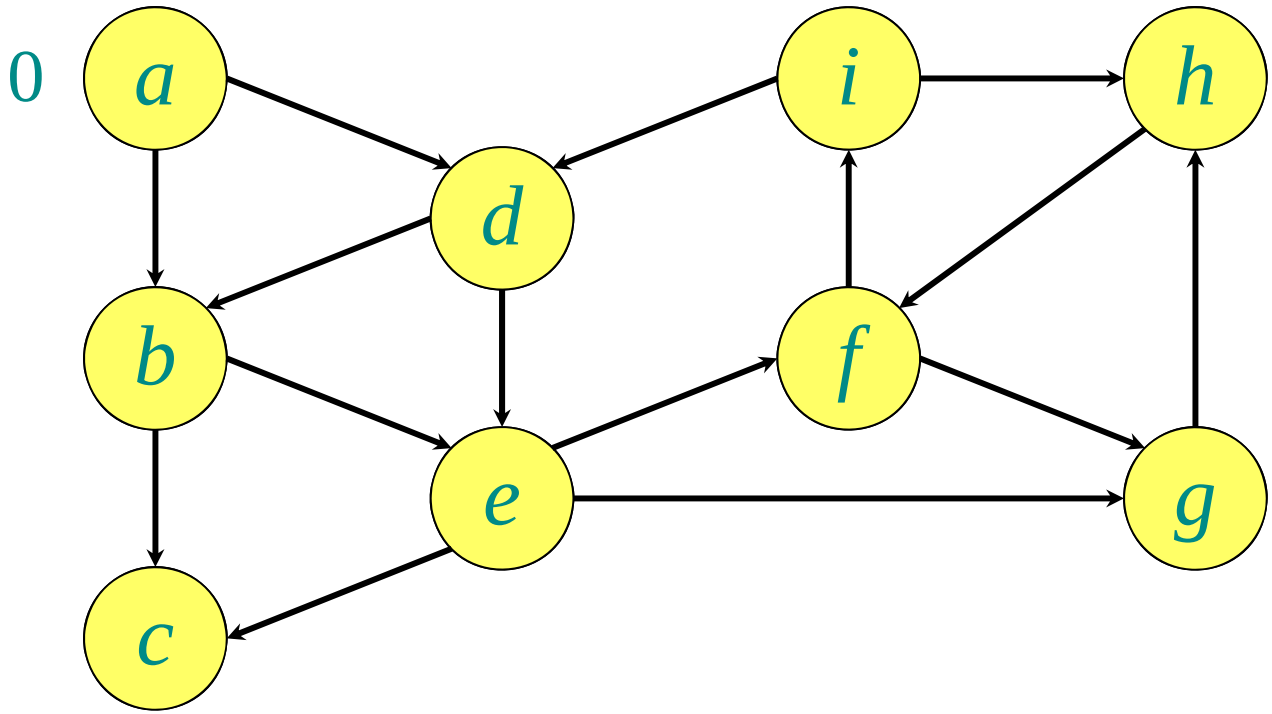


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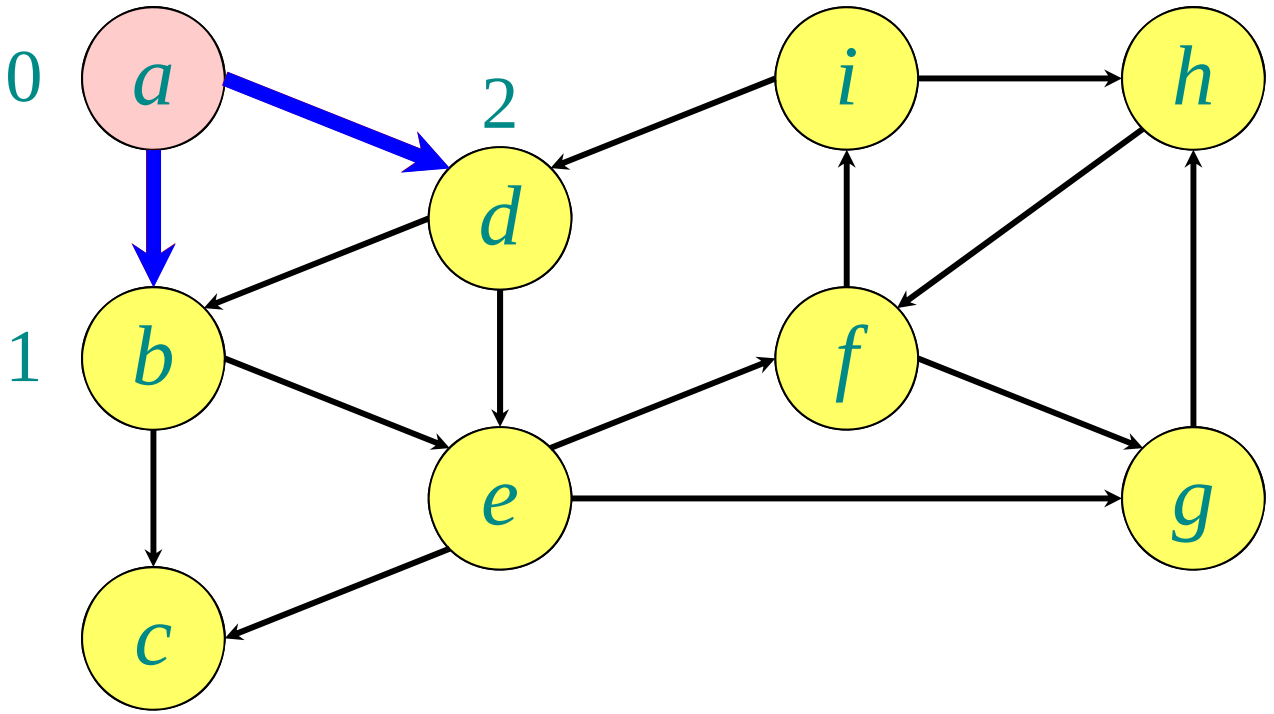


1 2

Q: a b d

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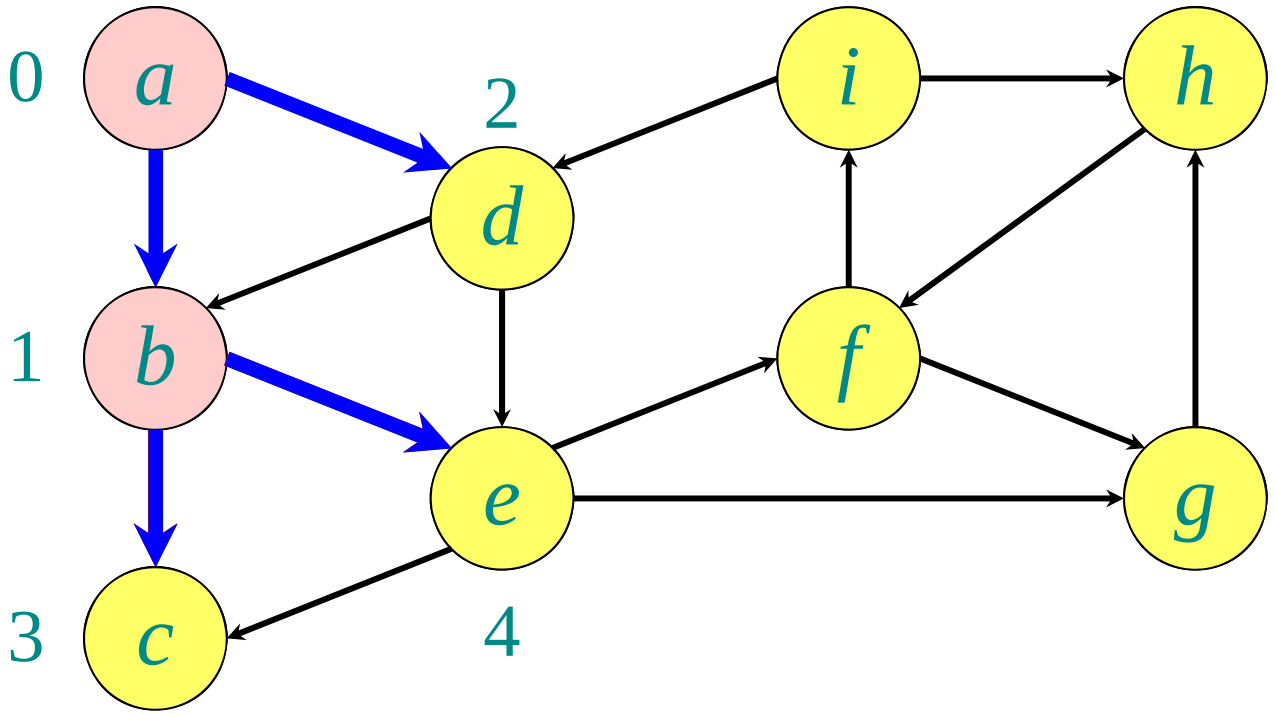


2 3 4

*Q:* *a* *b* *d* *c* *e*

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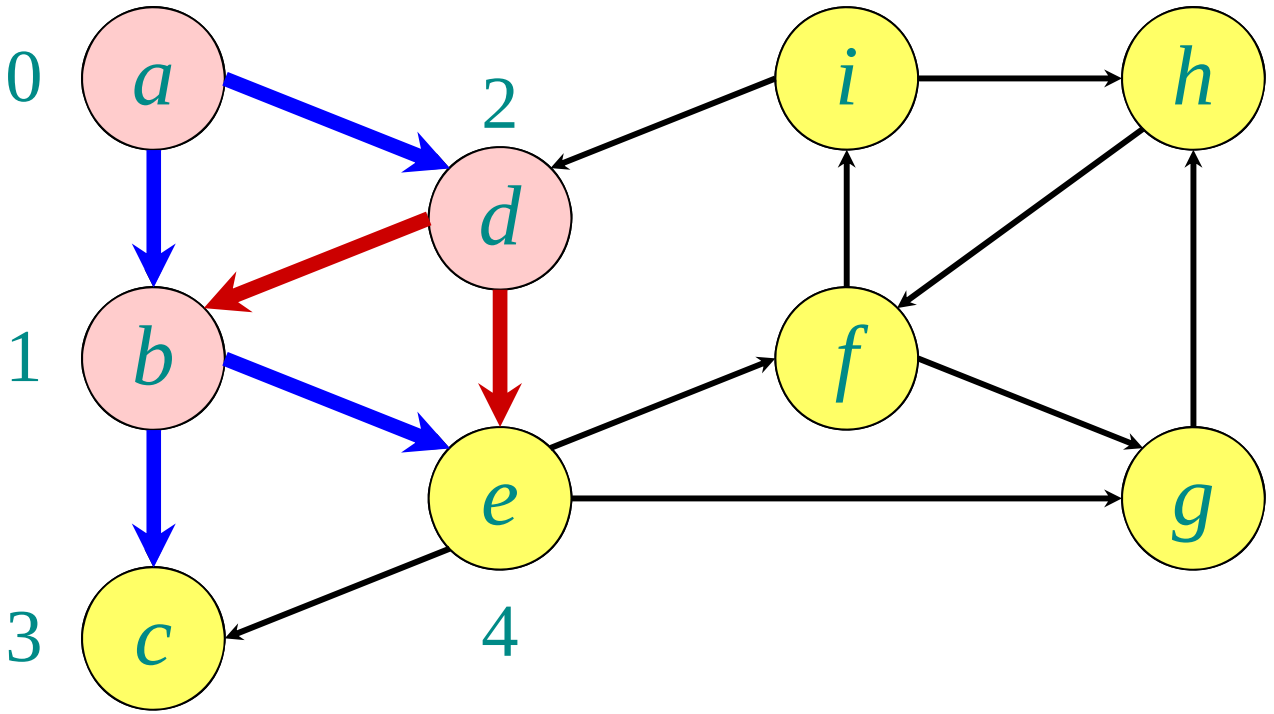


3 4

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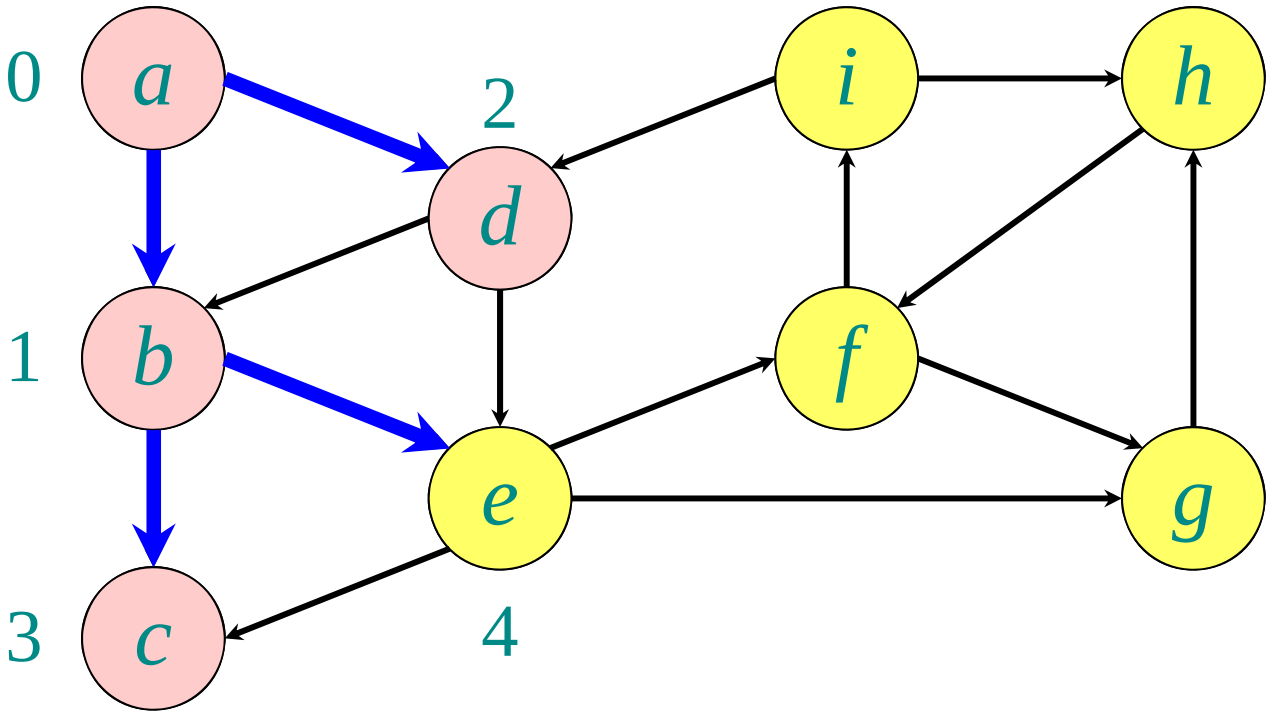


4

Q: a b d c e

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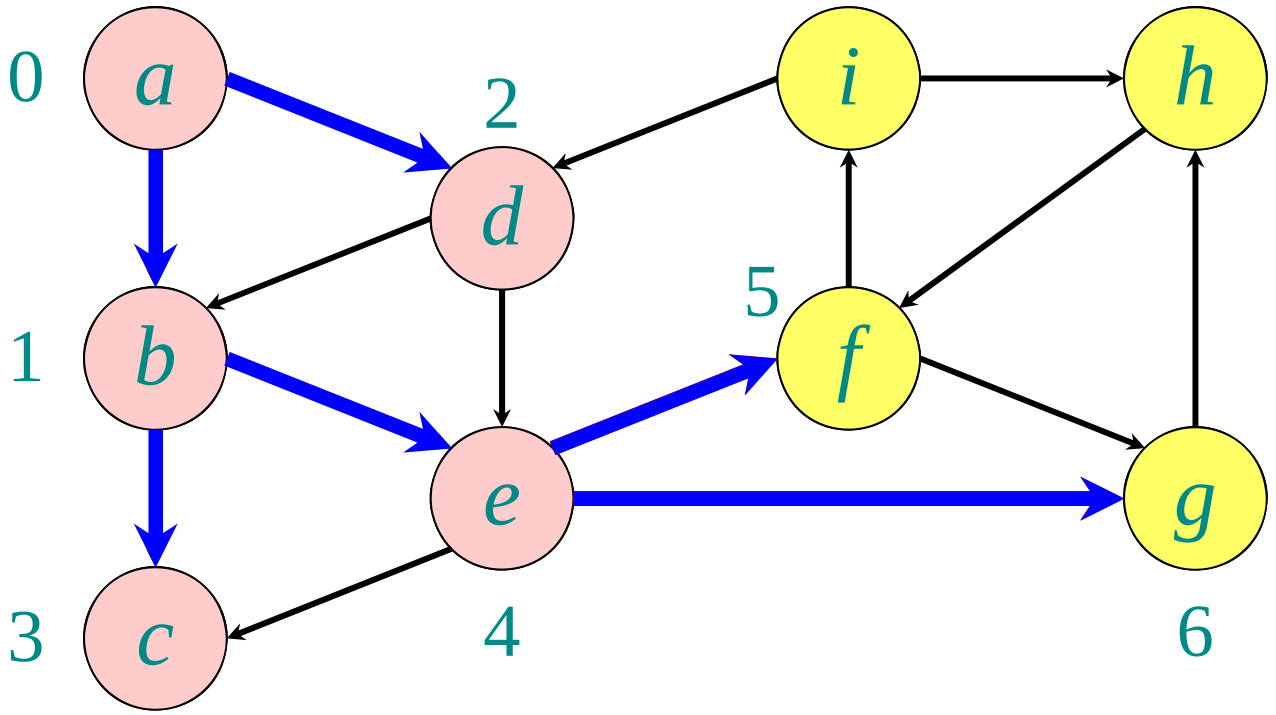


5 6

Q: a b d c e f g

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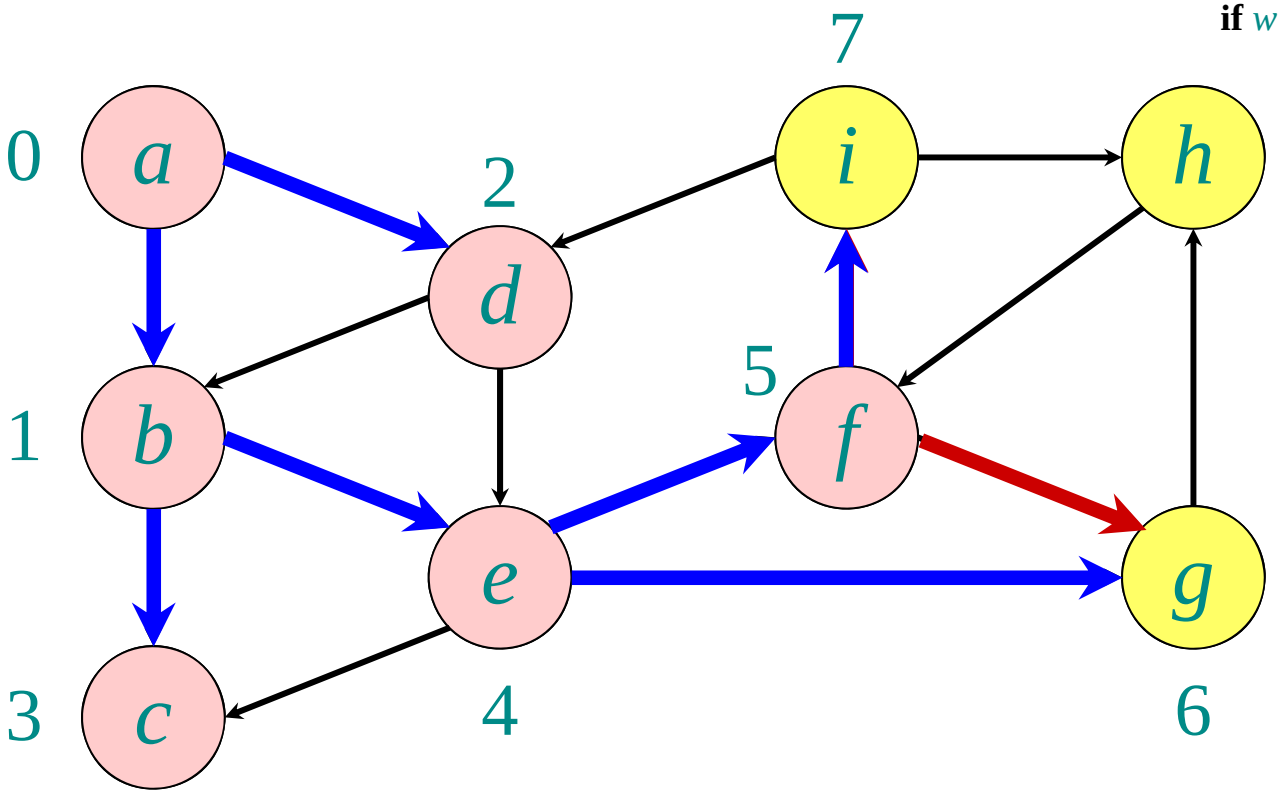


6 7

Q: a b d c e f g i

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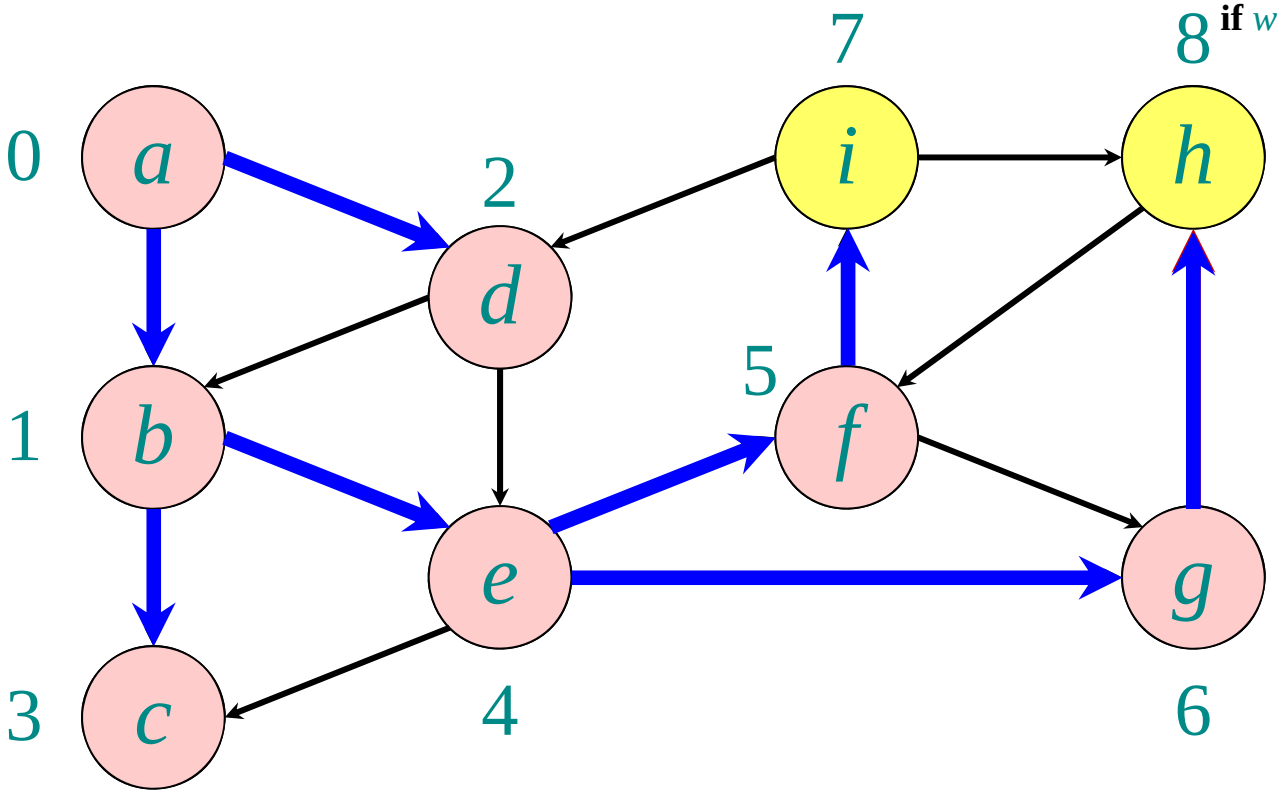


7 8

Q: a b d c e f g i h

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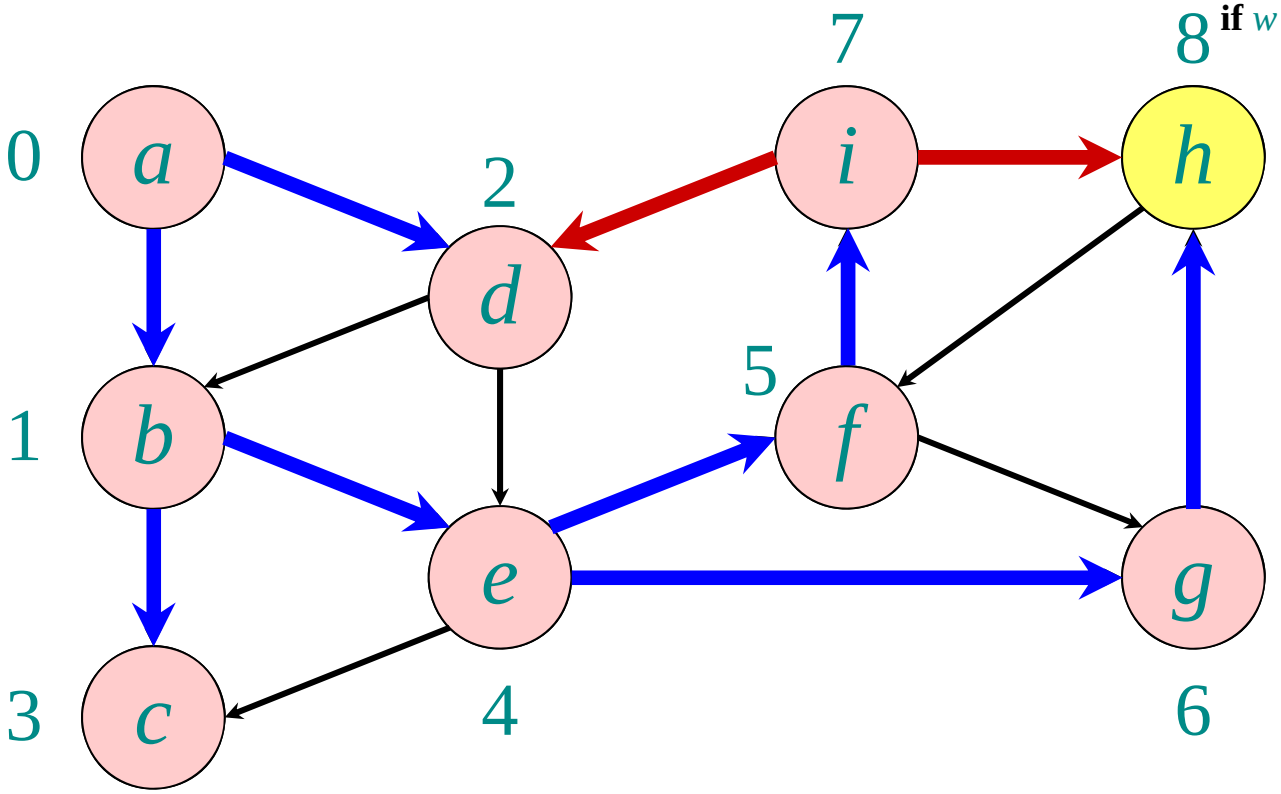


8

Q: a b d c e f g i h

# Example of breadth-first search

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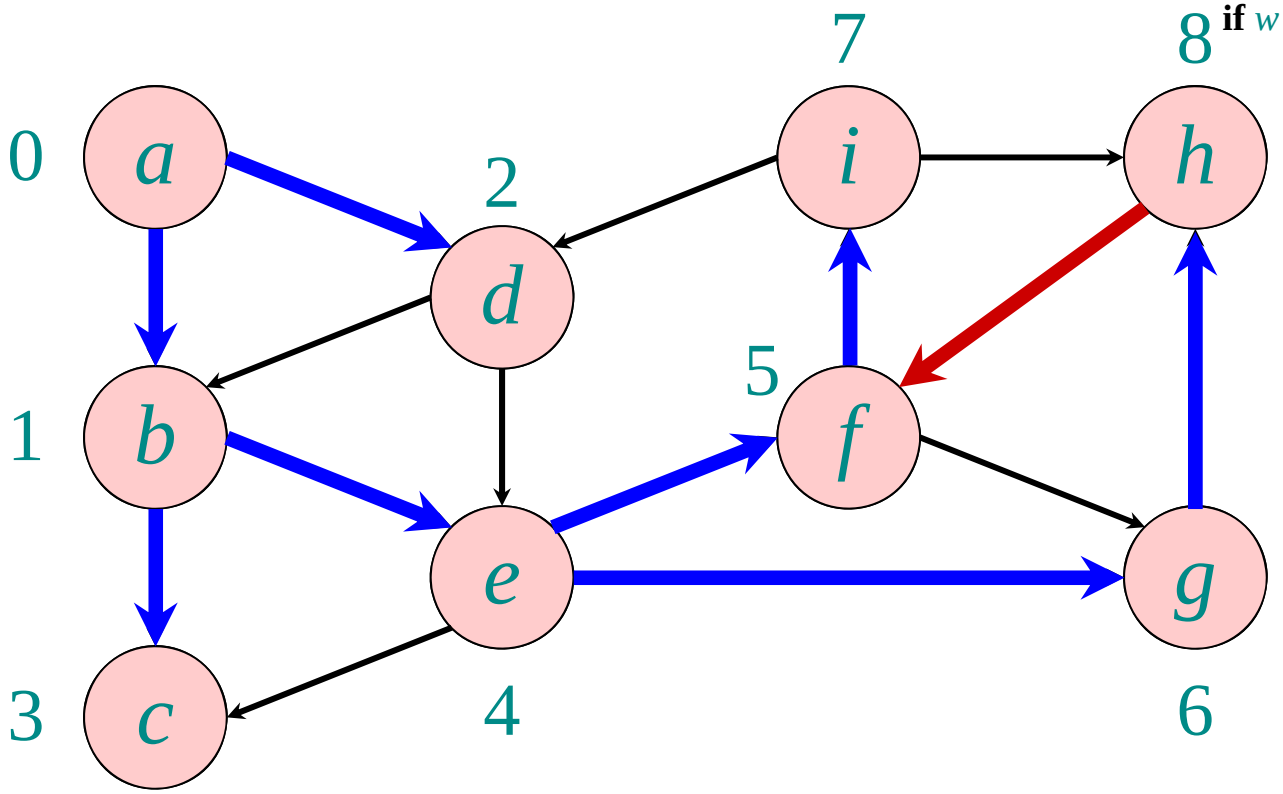


Q: a b d c e f g i h

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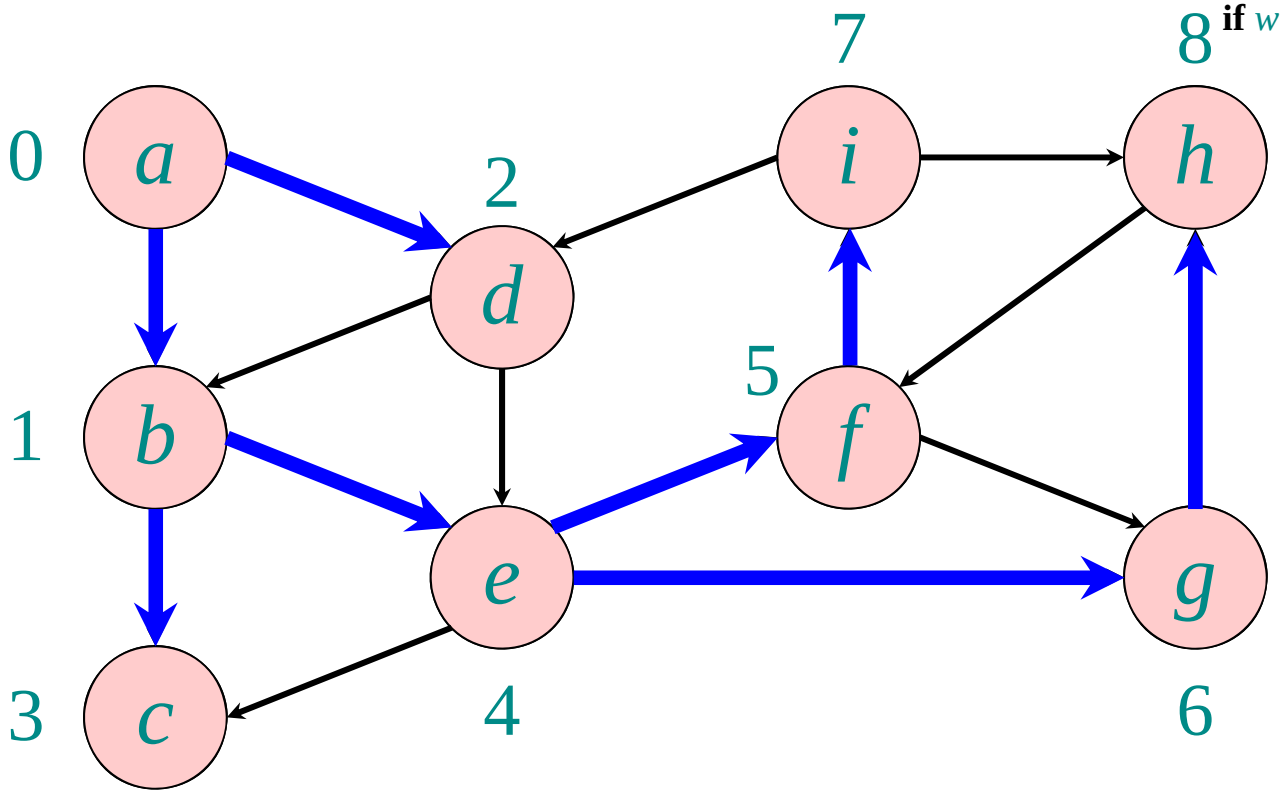
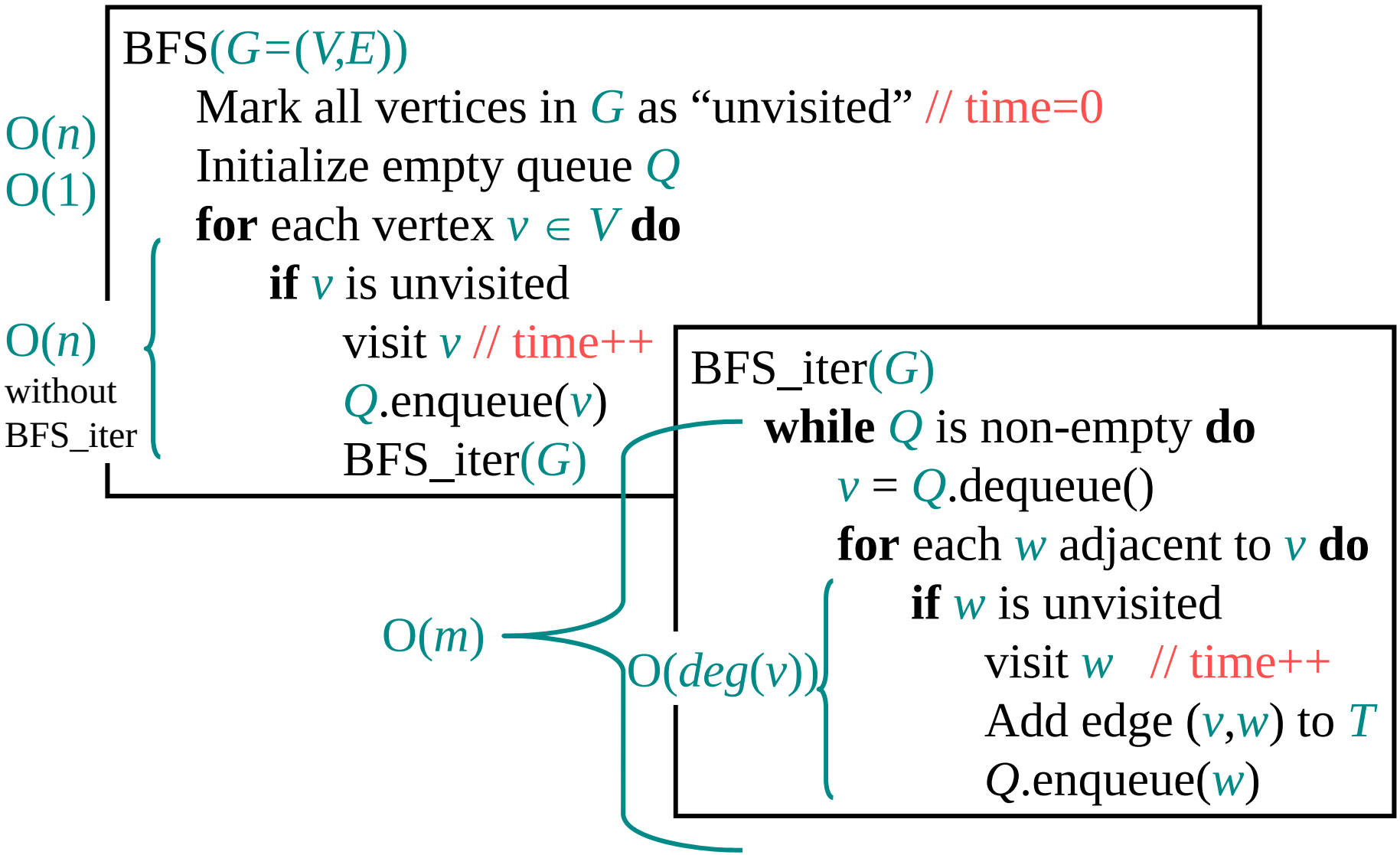



*Q:* *a b d c e f g i h*

# Breadth-First Search (BFS)



# BFS runtime

- Each vertex is marked as unvisited in the beginning $\Rightarrow$ $O(n)$ time
- Each vertex is marked at most once, enqueued at most once, and therefore dequeued at most once
- The time to process a vertex is proportional to the size of its adjacency list (its degree), since the graph is given in adjacency list representation

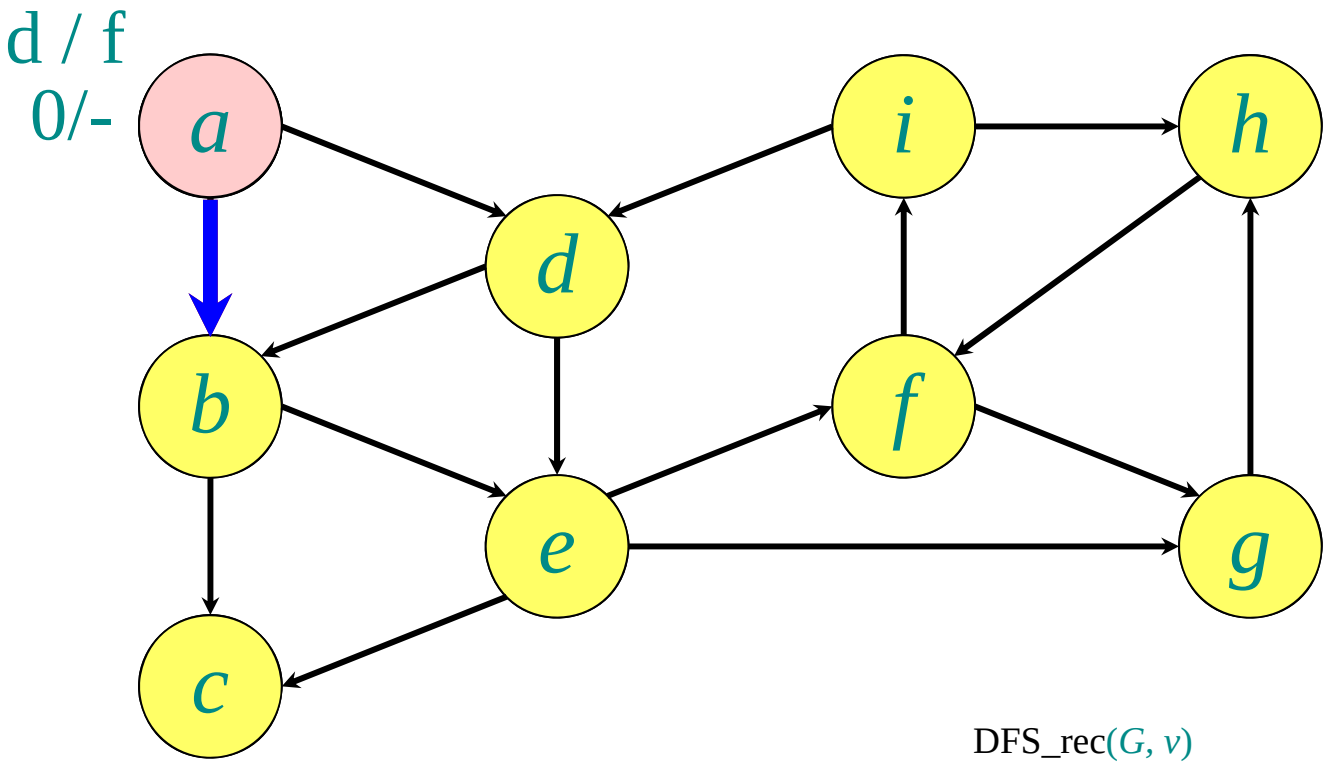
  $\Rightarrow$ $O(m)$ time
- Total runtime is $O(n+m) = O(|V| + |E|)$

# Depth-First Search (DFS)

```
DFS(G=(V,E))
    Mark all vertices in G as "unvisited"  // time=0
    for each vertex v ∈ V do
        if v is unvisited
            DFS_rec(G,v)
```

```
DFS_rec(G, v)
    mark v as "visited" // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    mark v as "finished" // f[v]=++time
```

# Example of depth-first search




π: a b c d e f g h i

- a a

Store edges in predecessor array

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DFS_rec(G, v)
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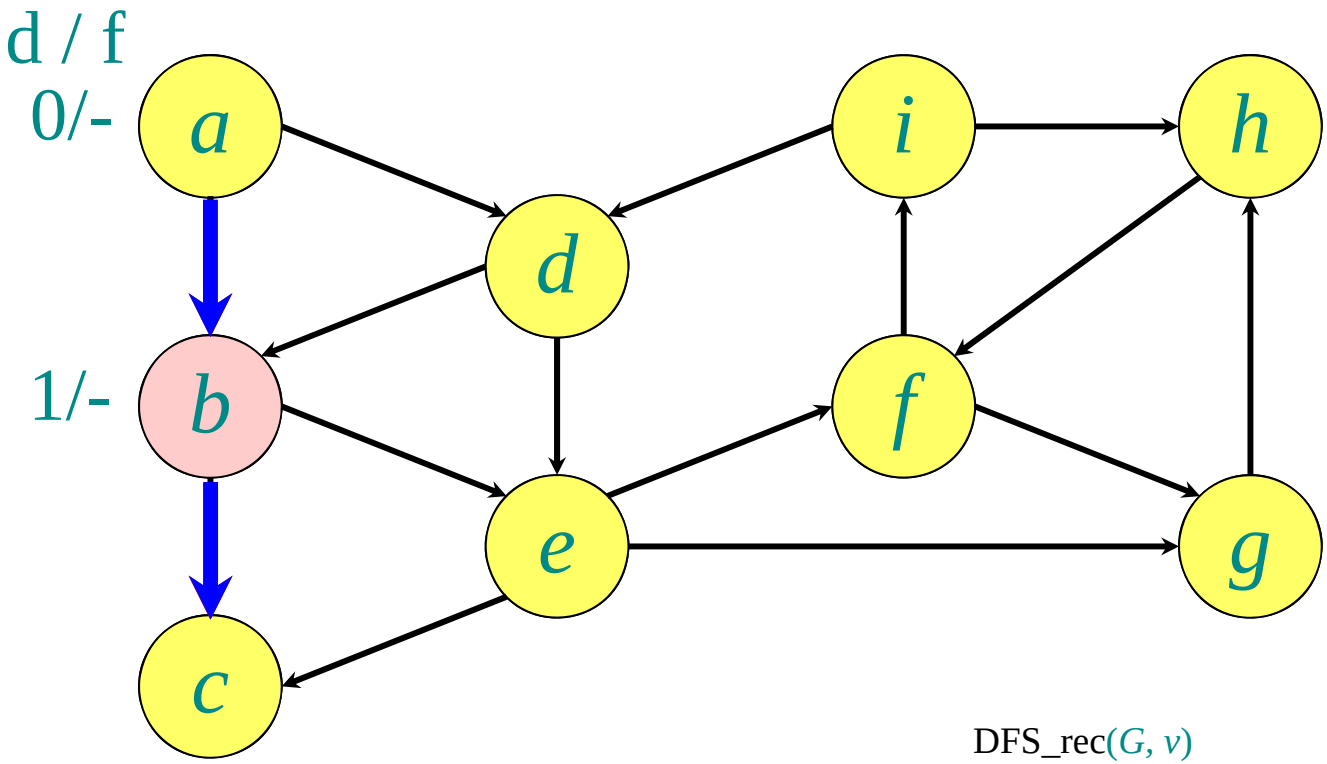


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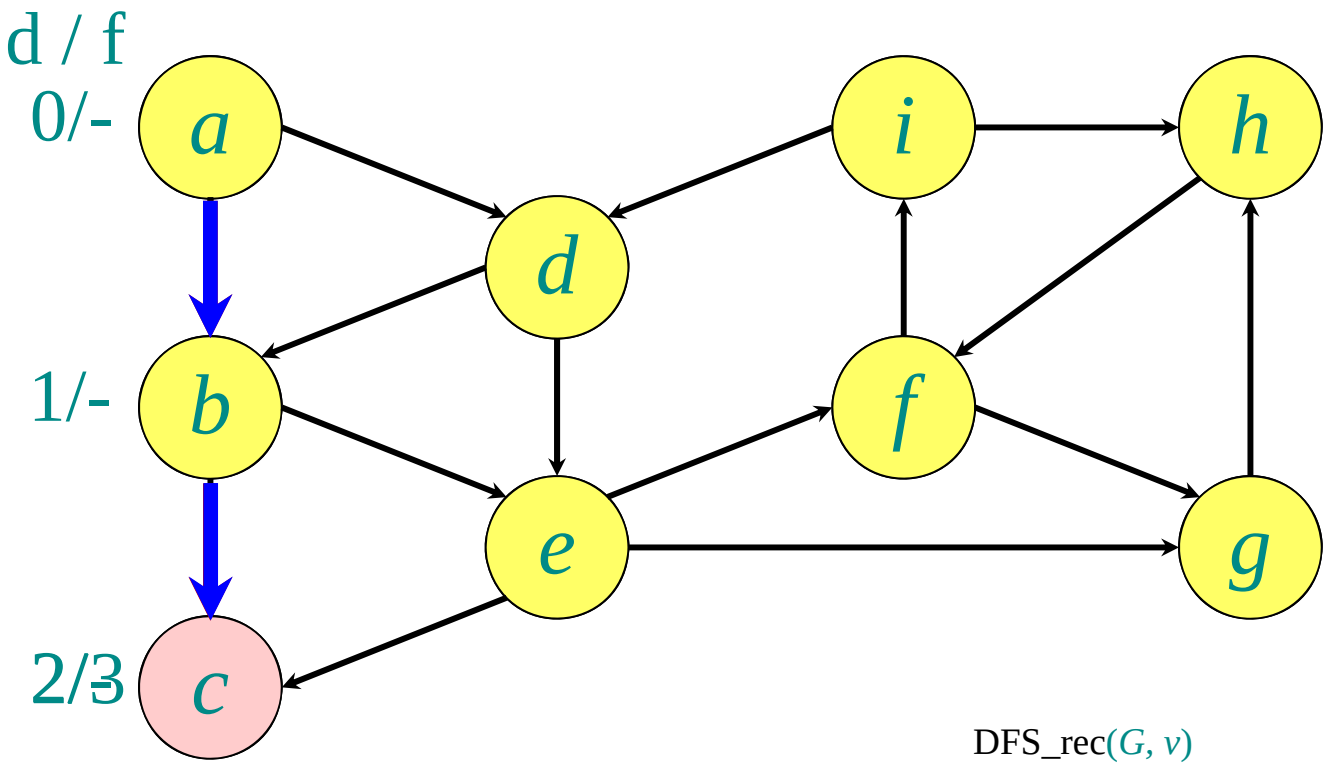


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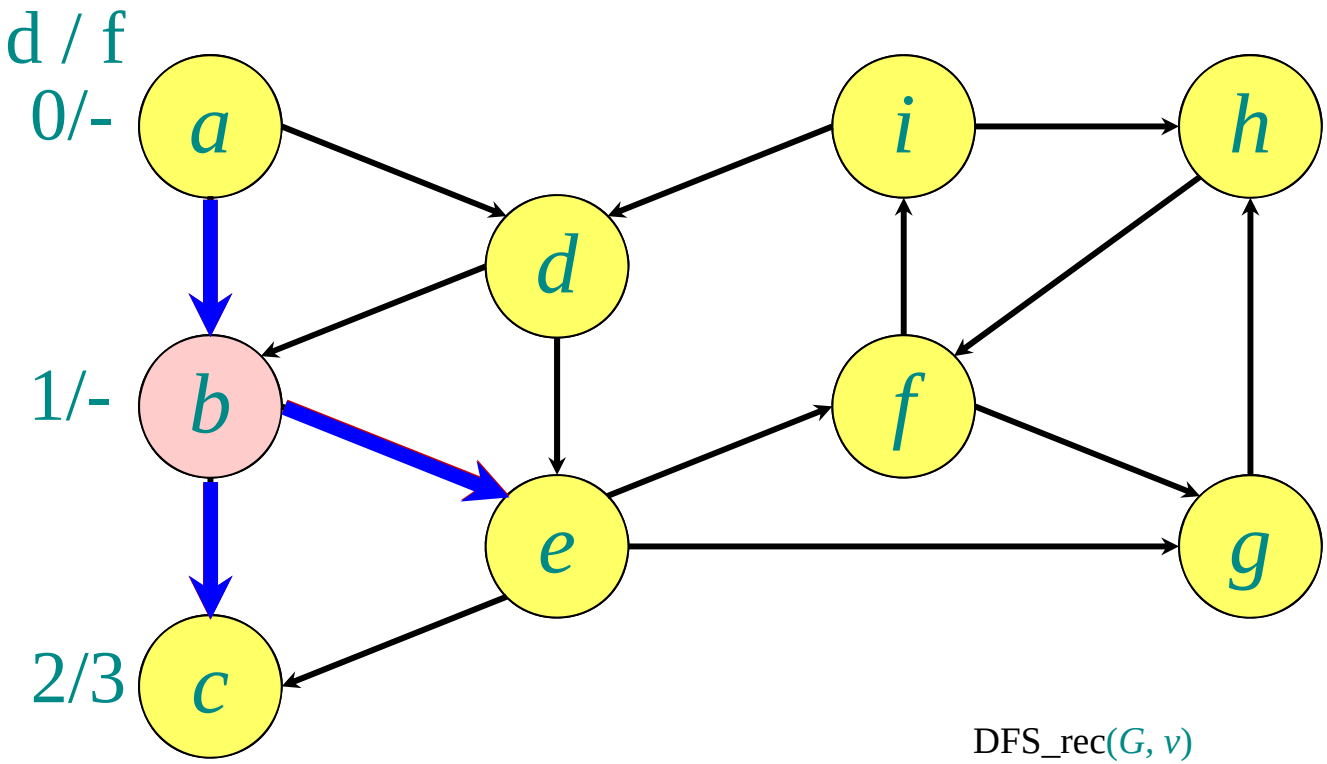


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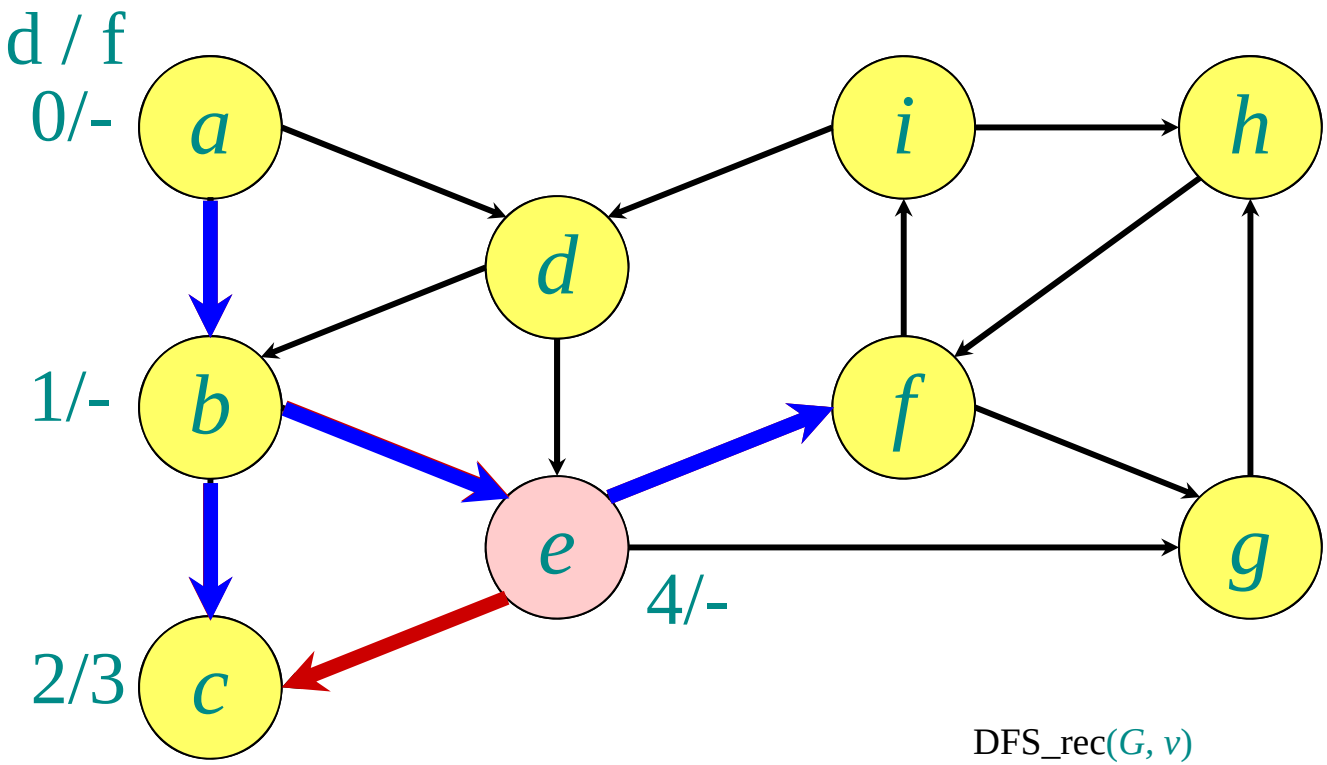


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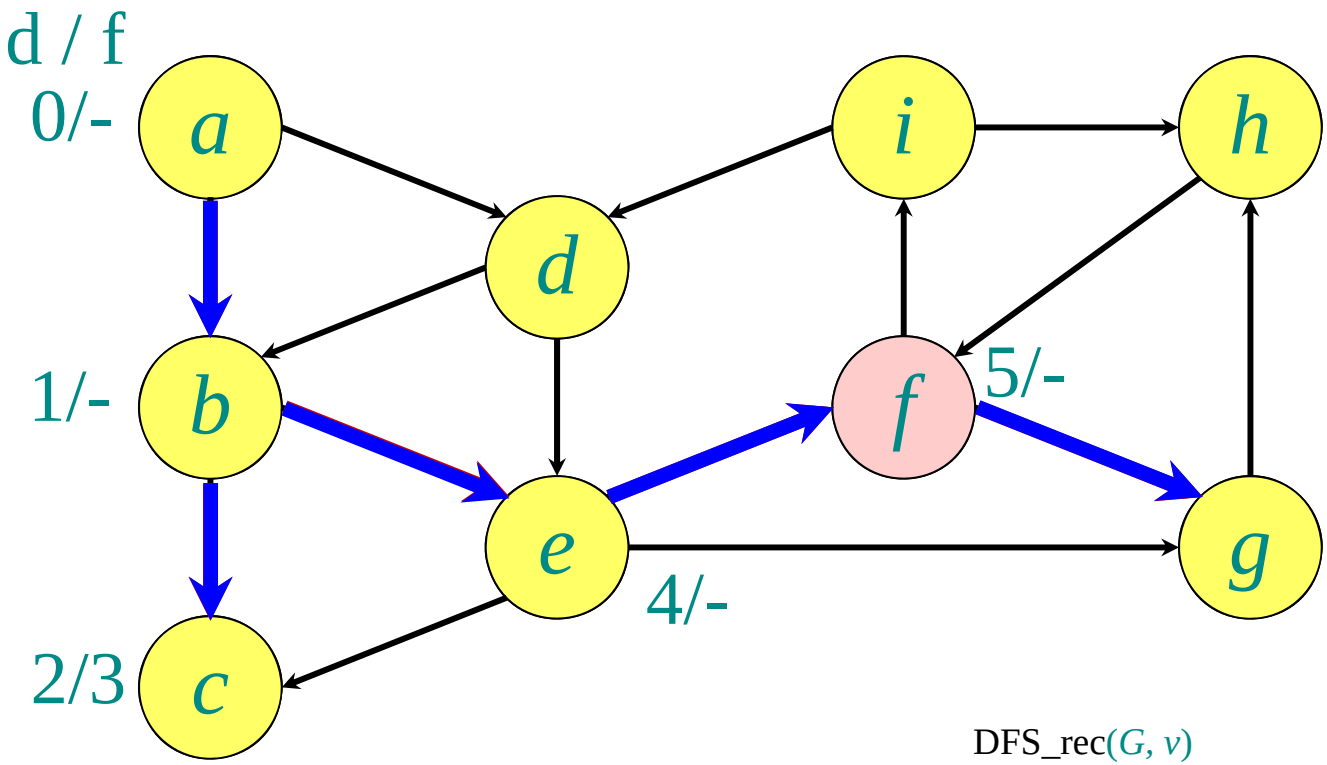


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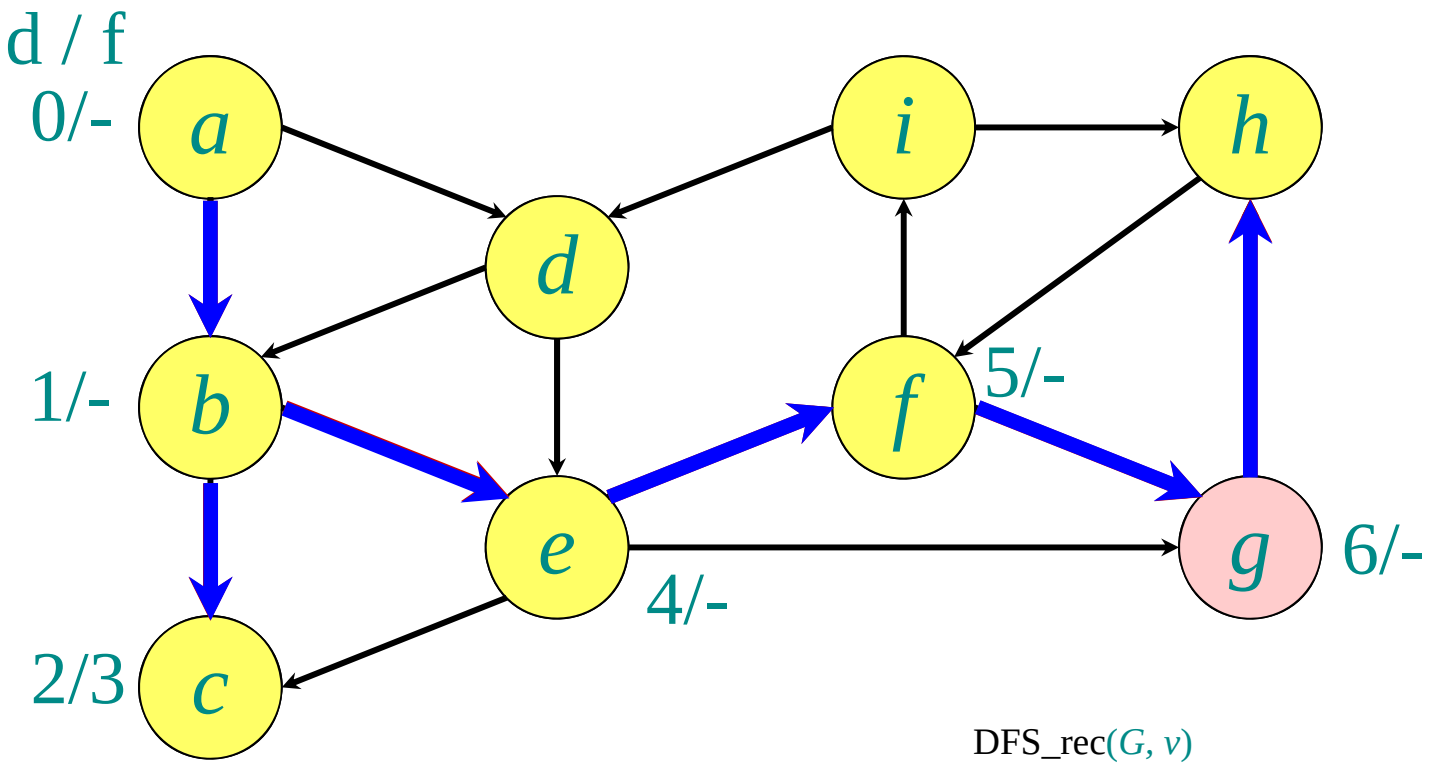


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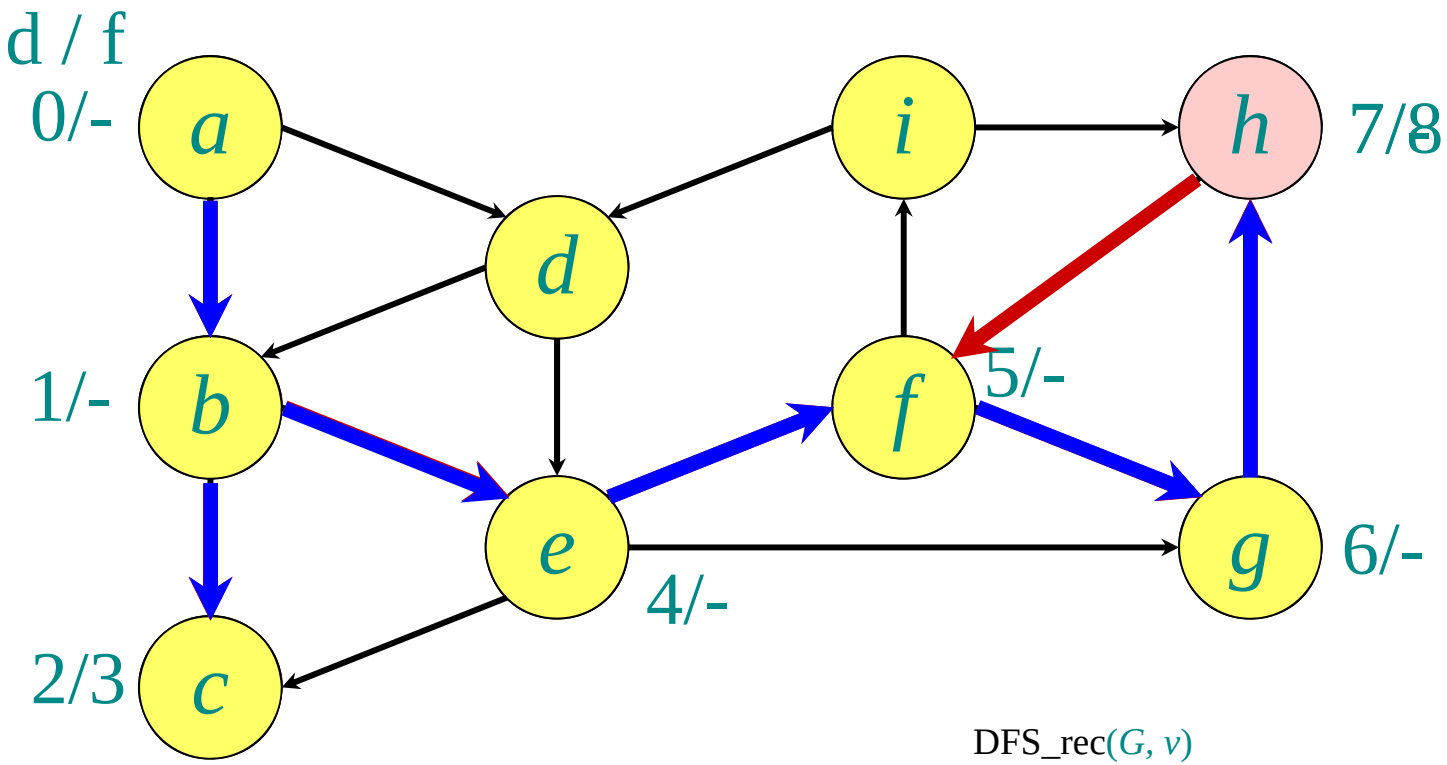


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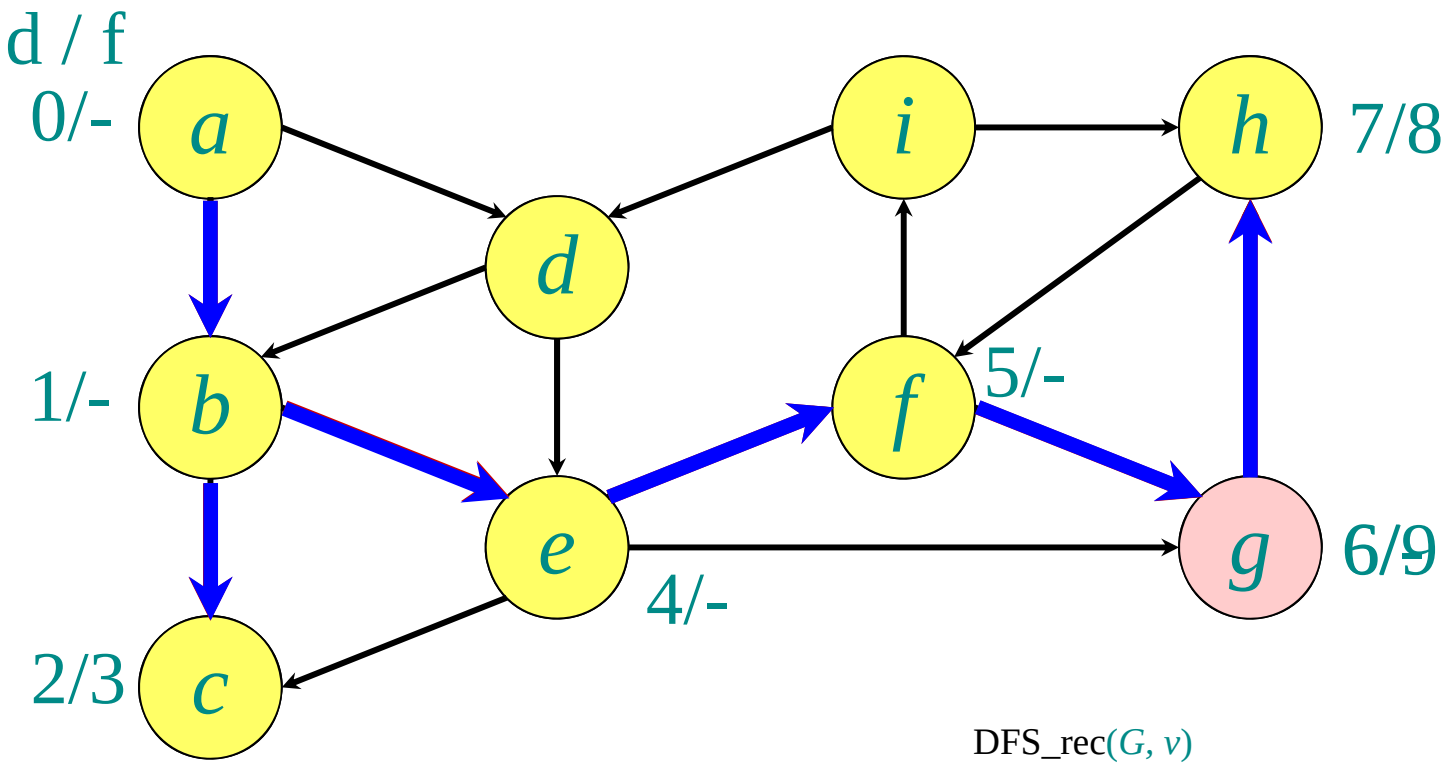


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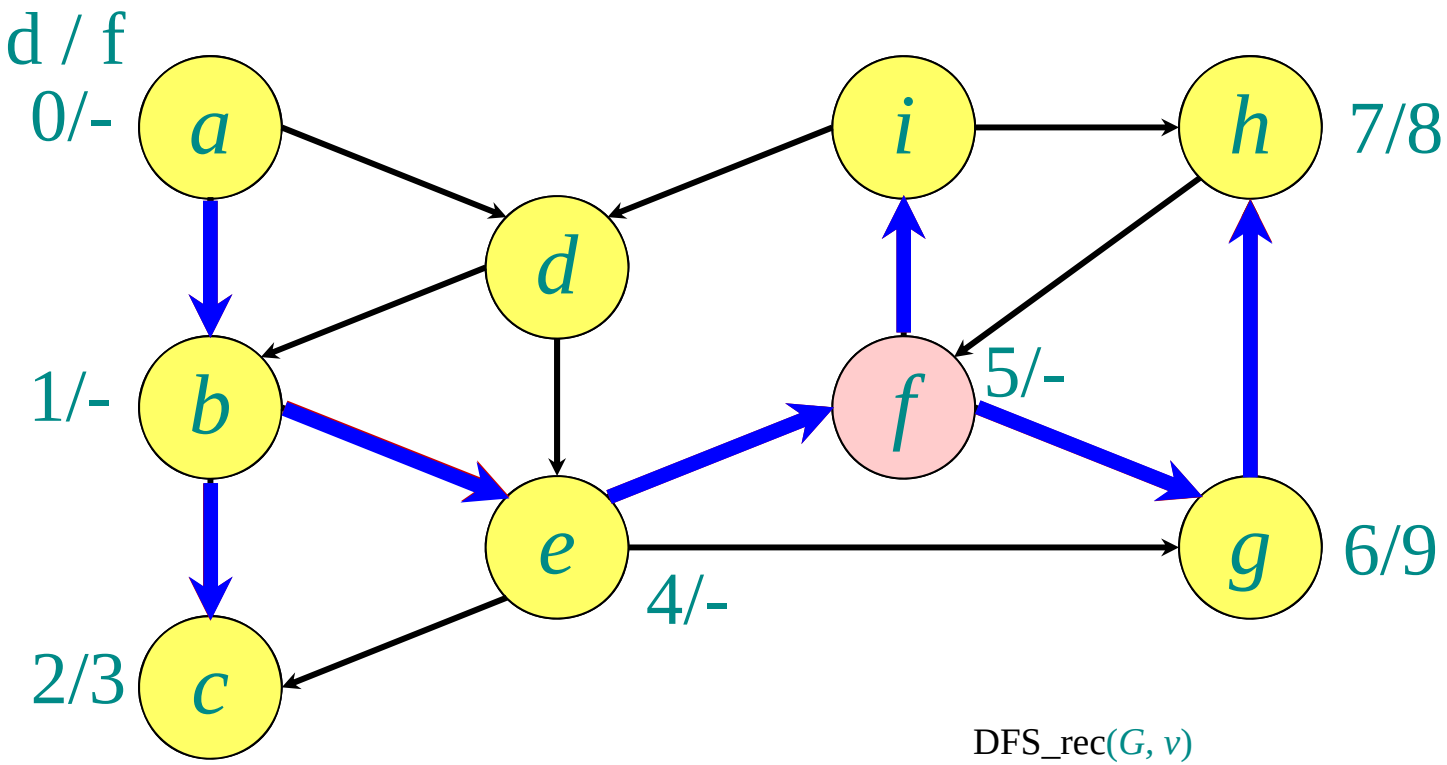


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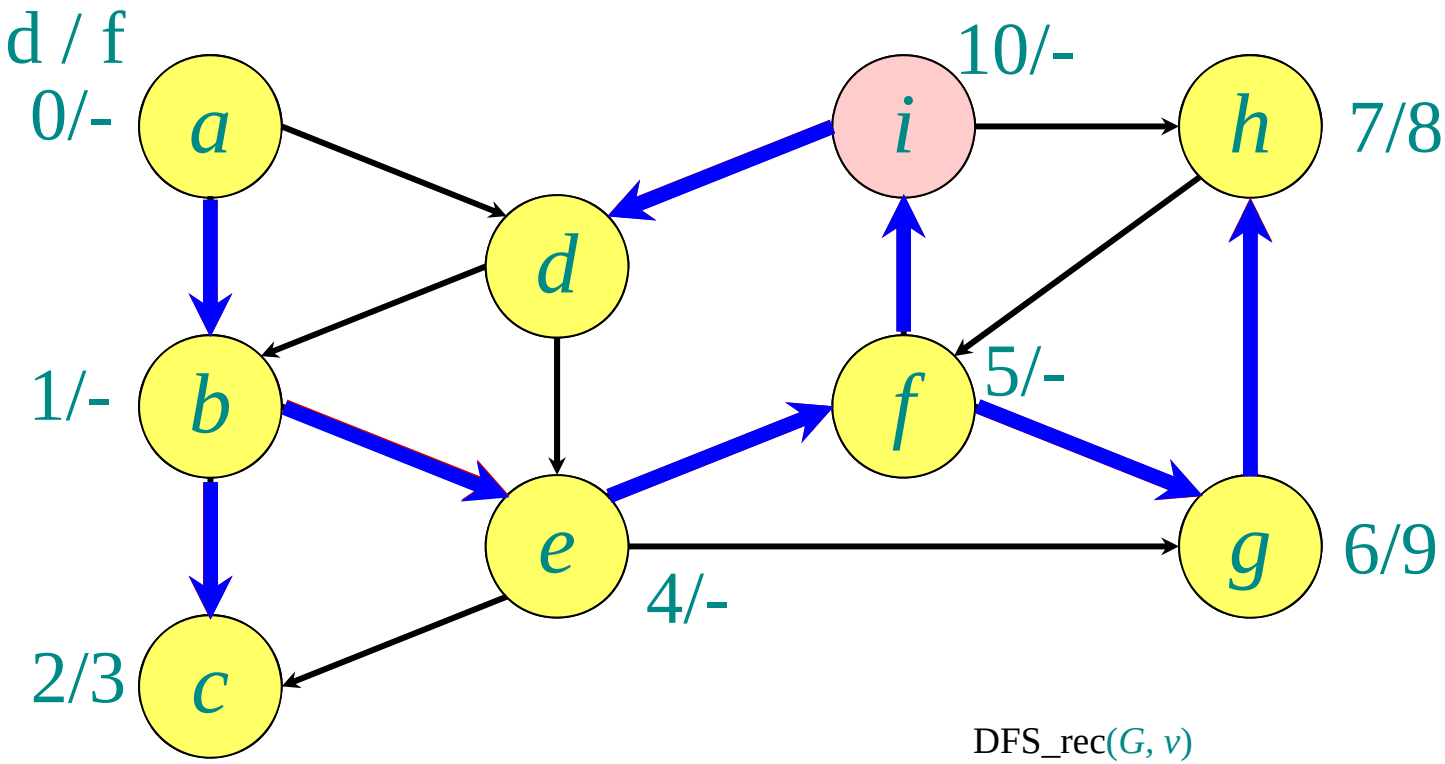


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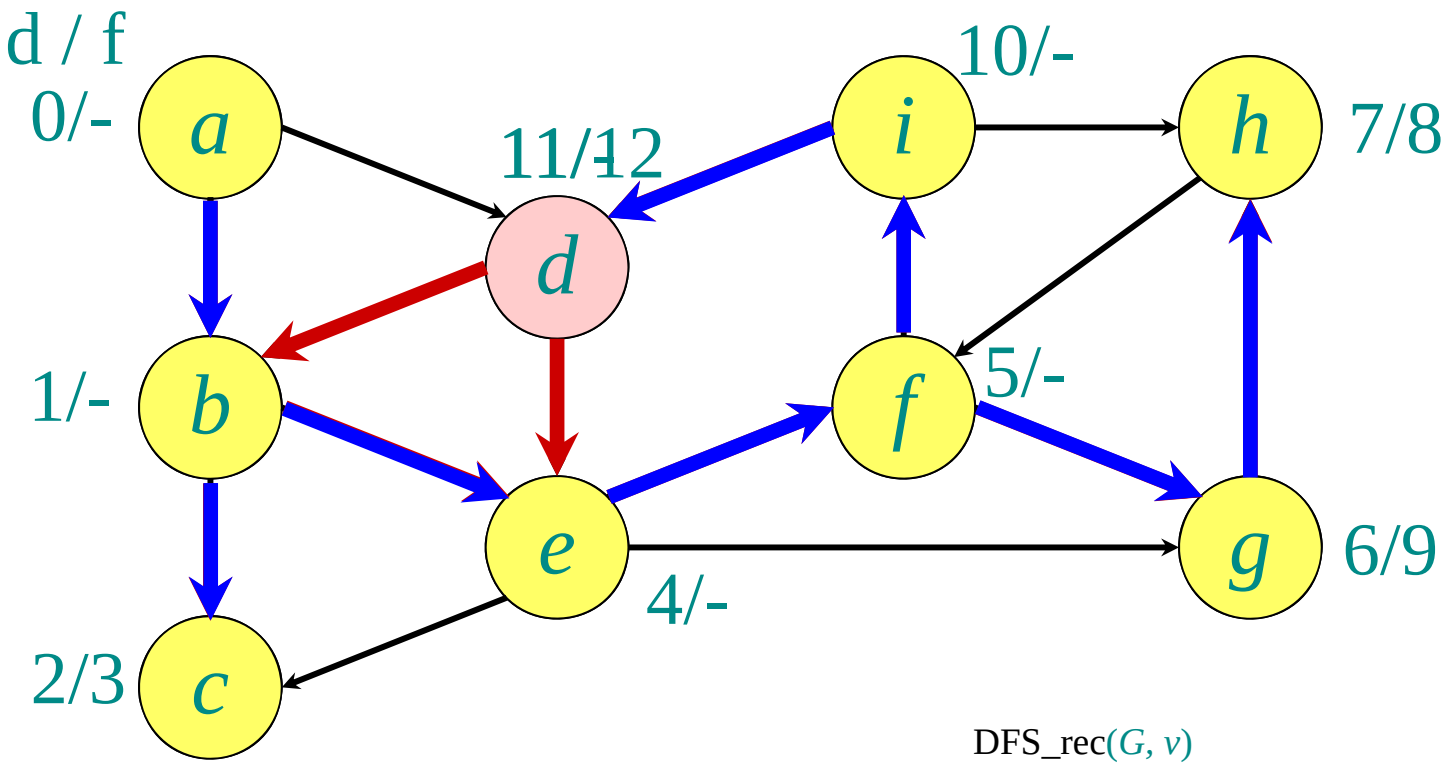


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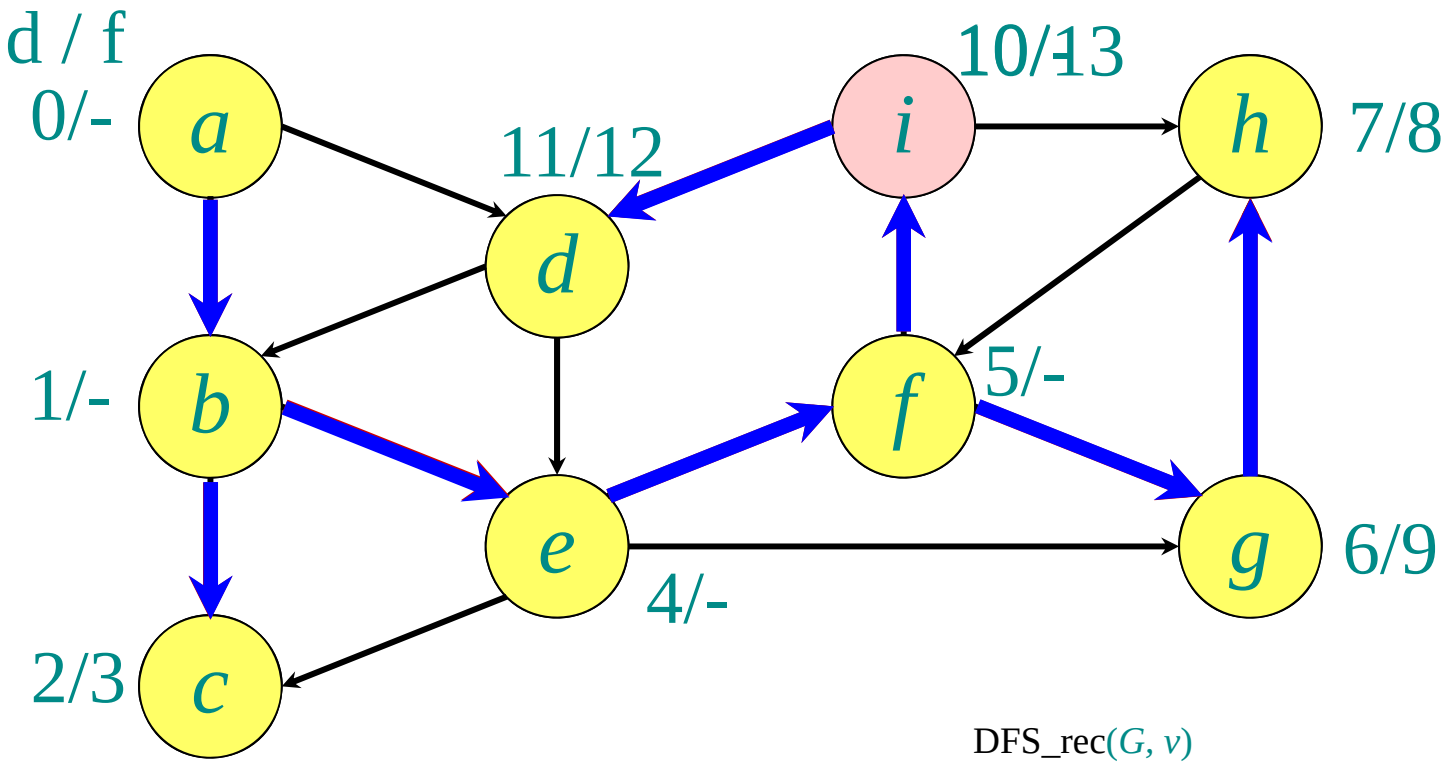


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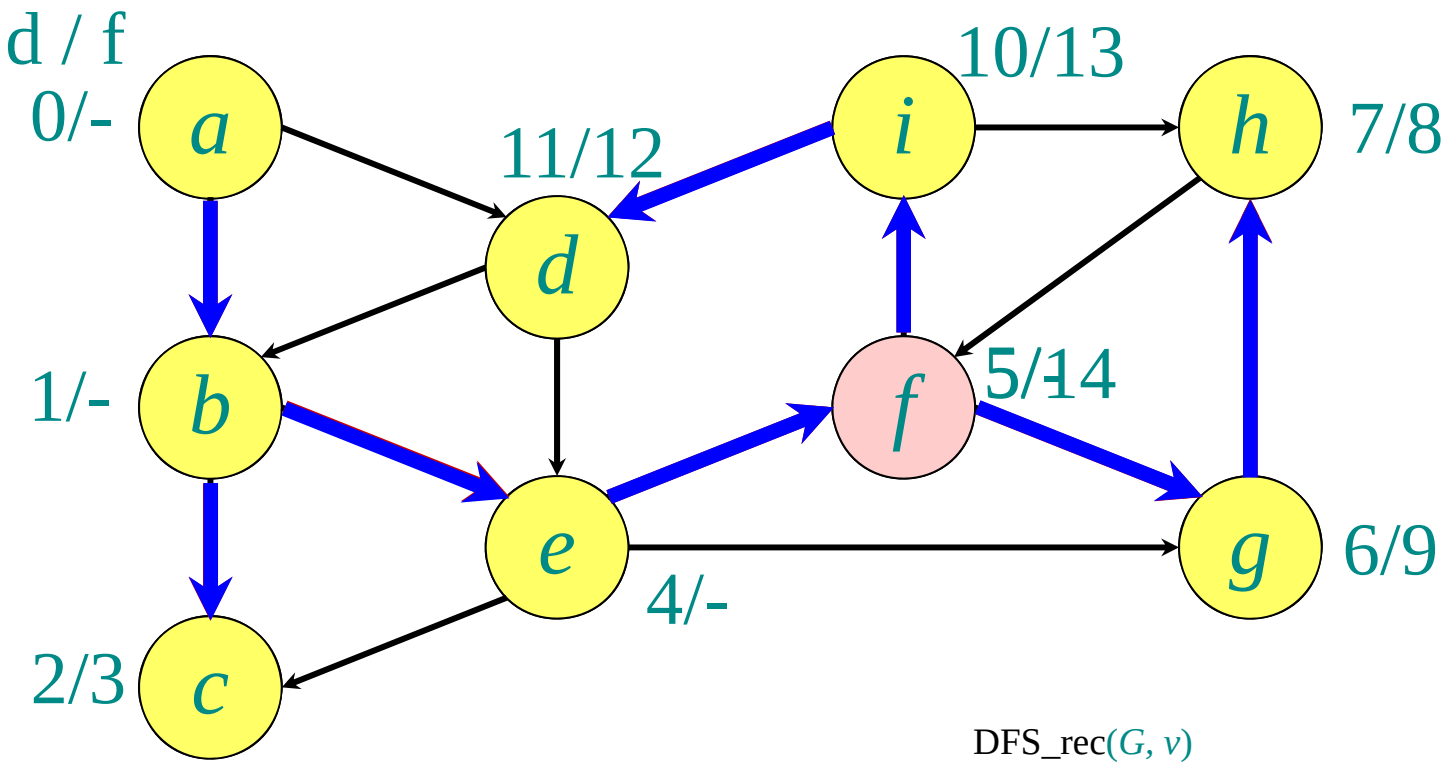


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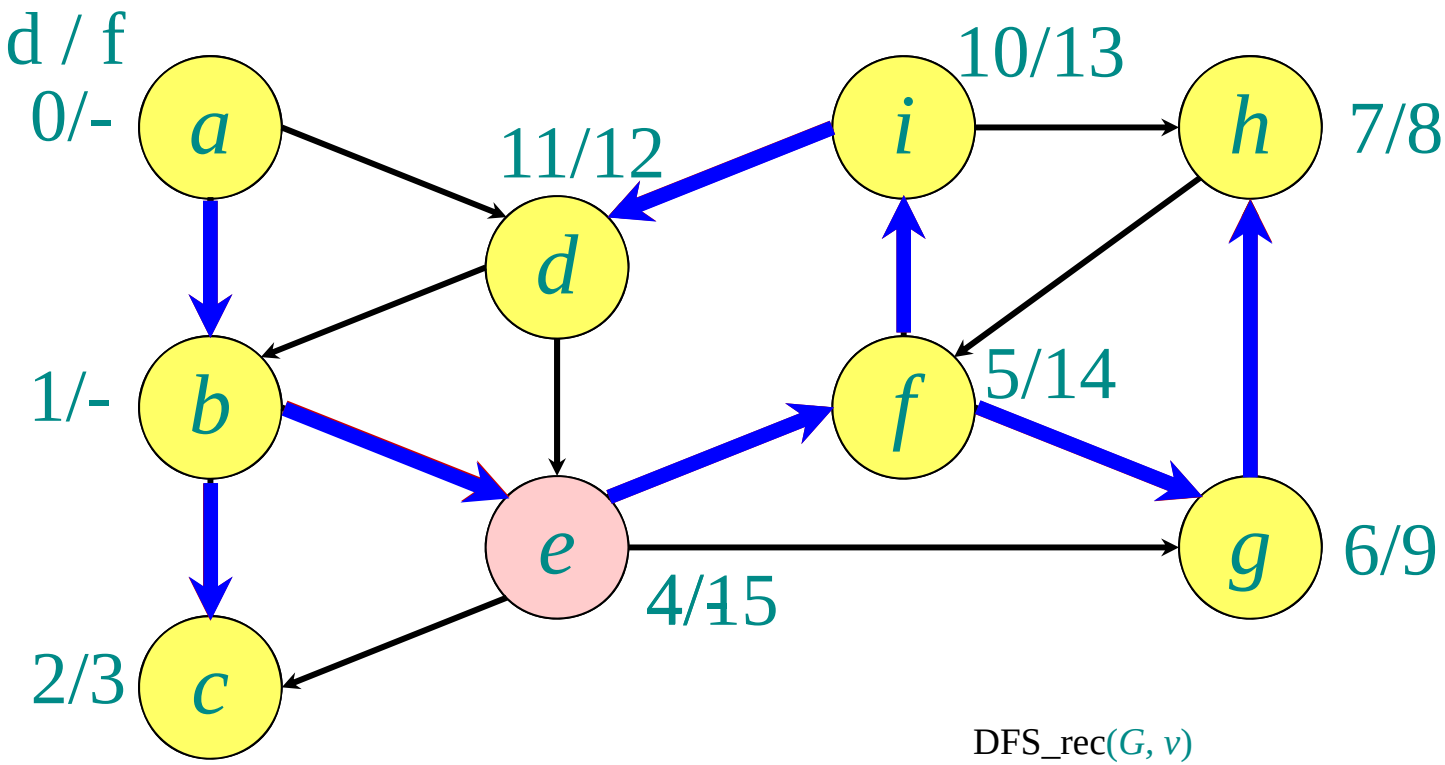
# Example of depth-first search




π: a b c d e f g h i
- a b i b e f g f

Store edges in predecessor array

```
DFS_rec(G, v)
    mark v as "visited" // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    mark v as "finished" // f[v]=++time
```

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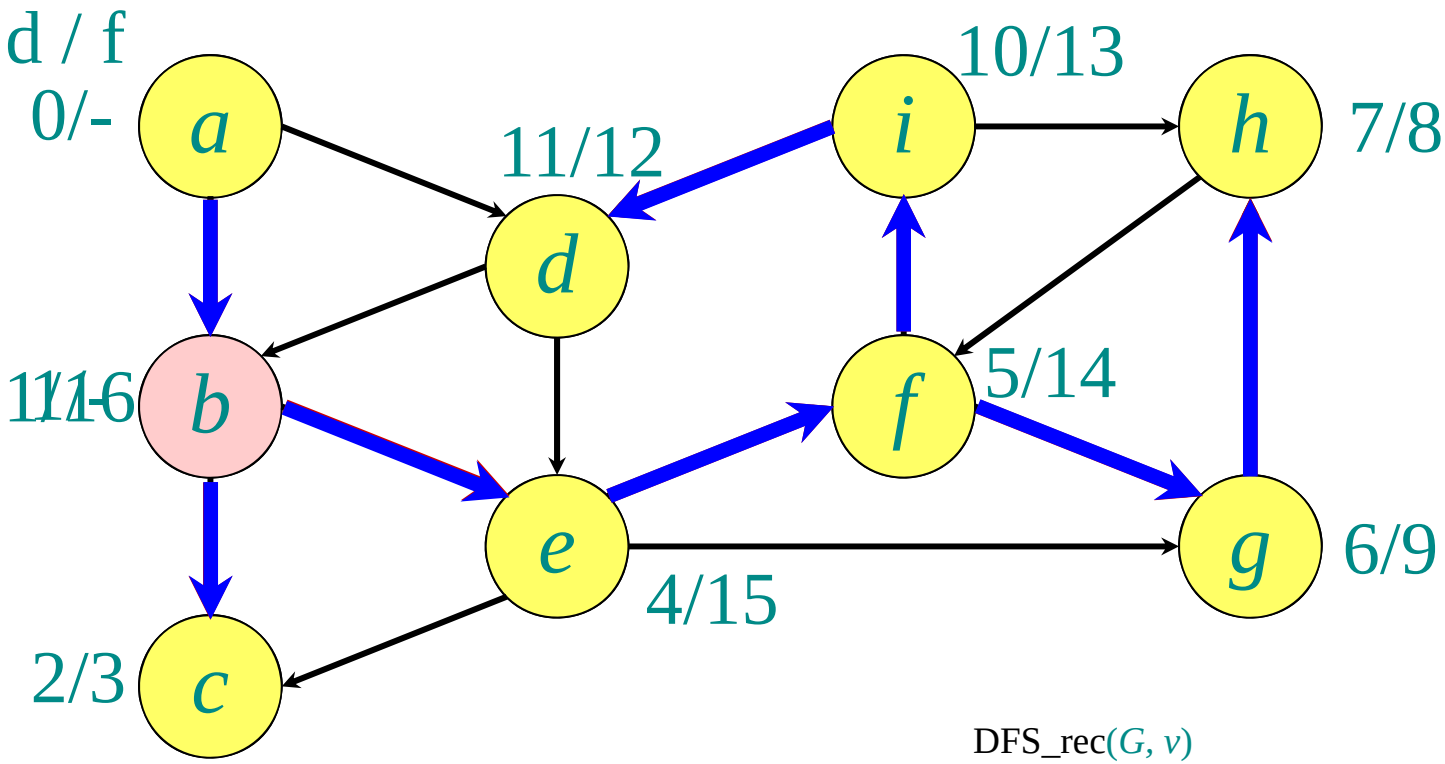


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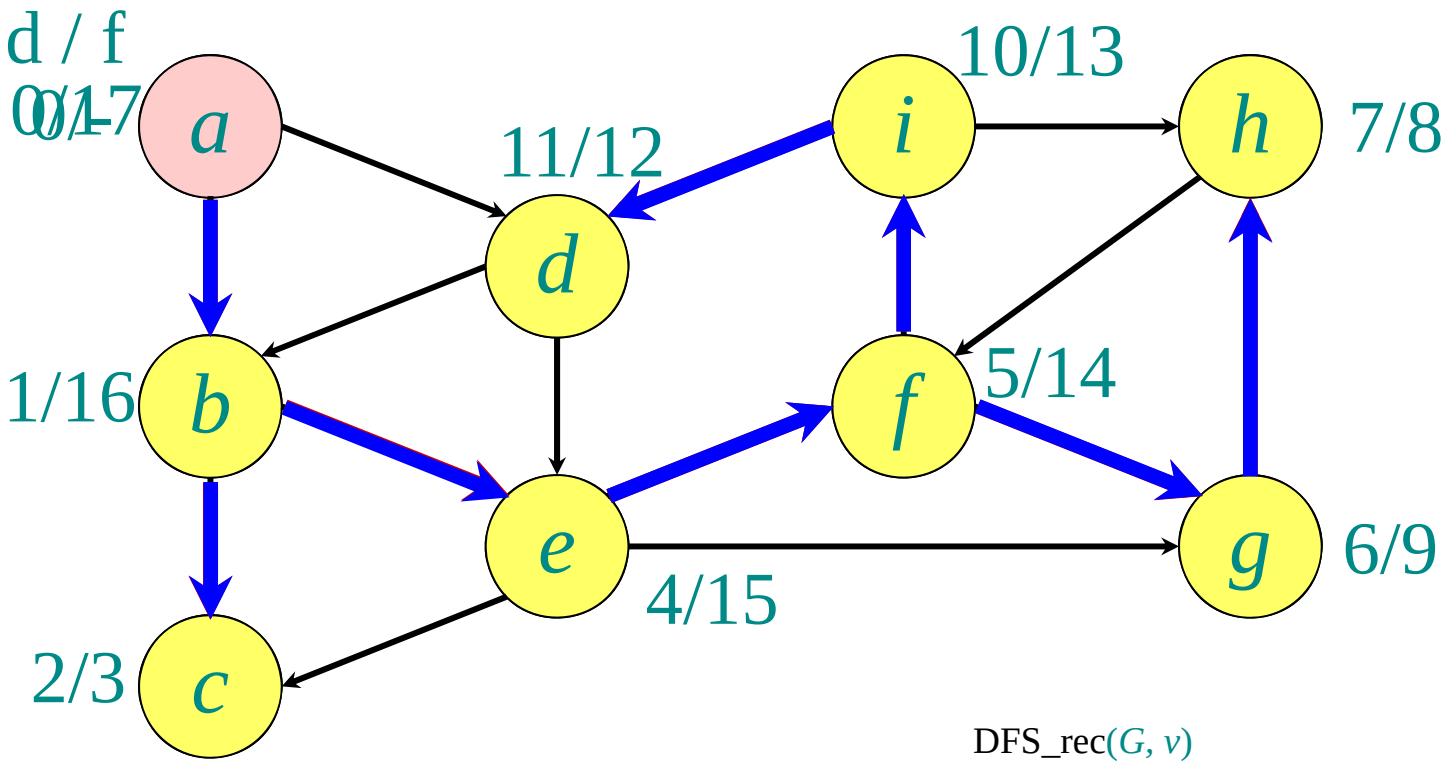


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            Add edge (v,w) to tree T
            DFS_rec(G,w)
    mark v as "finished" // f[v]=++time
```

# Depth-First Search (DFS)

```
DFS(G=(V,E))
    Mark all vertices in G as "unvisited"  // time=0
    for each vertex v ∈ V do
        if v is unvisited
            DFS_rec(G,v)
```

O(*n*)

O(*n*) without DFS_rec

```
DFS_rec(G, v)
    mark v as "visited" // d[v]=++time
    for each w adjacent to v do
        if w is unvisited
            Add edge (v,w) to tree T
            DFS_rec(G,w)
    mark v as "finished" // f[v]=++time
```

O(1)

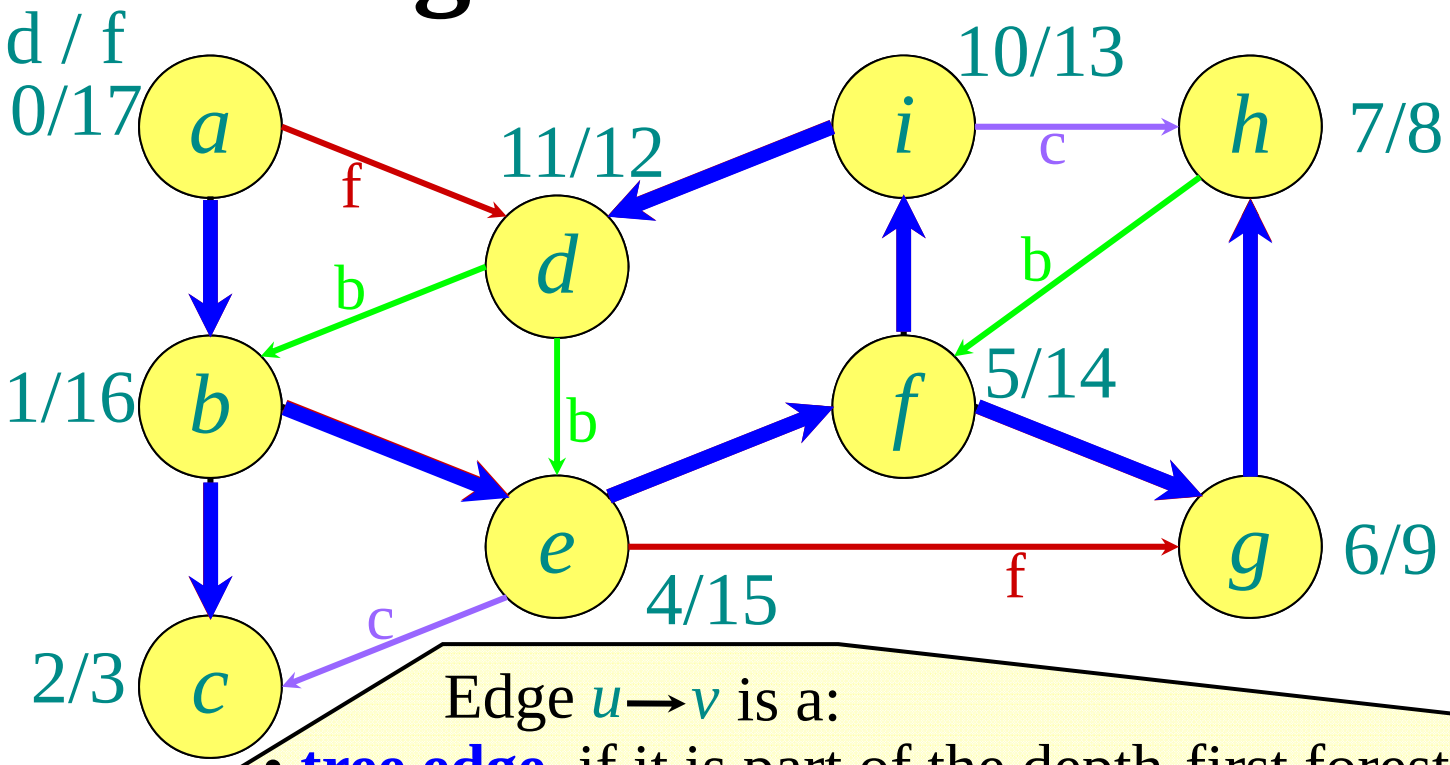
O(*deg*(*v*)) without recursive call

⇒ With Handshaking Lemma, all recursive calls are O(*m*), for a total of O(*n* + *m*) runtime

# DFS runtime

- Each vertex is visited at most once $\Rightarrow$ $O(n)$ time
- The body of the **for** loops (except the recursive call) take constant time per graph edge
- All **for** loops take $O(m)$ time
- Total runtime is $O(n+m) = O(|V| + |E|)$

# DFS edge classification




Edge $u \rightarrow v$ is a:

- **tree edge**, if it is part of the depth-first forest.
- back edge, if $u$ connects to an ancestor $v$ in a depth-first tree. It holds $d(u) > d(v)$ and $f(u) < f(v)$.
- forward edge, if it connects $u$ to a descendant $v$ in a depth-first tree. It holds $d(u) < d(v)$.
- cross edge, if it is any other edge. It holds $d(u) > d(v)$ and $f(u) > f(v)$.

# Paths, Cycles, Connectivity

Let $G=(V,E)$ be a directed (or undirected) graph

- A **path** from $v_1$ to $v_k$ in $G$ is a sequence of vertices $v_1, v_2,\ldots,v_k$ such that $(v_i,v_{\{i+1\}})\in E$ (or $\{v_i,v_{\{i+1\}}\} \in E$ if $G$ is undirected) for all $i\in\{1,\ldots,k\text{-}1\}$.
- A path is **simple** if all vertices in the path are distinct.
- A path $v_1, v_2,\ldots,v_k$ forms a **cycle** if $v_1=v_k$ .
- A graph with no cycles is **acyclic**.
  - An undirected acyclic graph is called a **tree**. (Trees do not have to have a root vertex specified.)
  - A directed acyclic graph is a **DAG**. (A DAG can have undirected cycles if the direction of the edges is not considered.)
- An undirected graph is **connected** if every pair of vertices is connected by a path. A directed graph is **strongly connected** if for every pair $u,v\in V$ there is a path from $u$ to $v$ and there is a path from $v$ to $u$.
- The **(strongly) connected components** of a graph are the equivalence classes of vertices under this reachability relation.

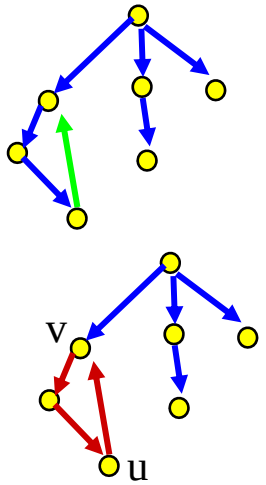
# DAG Theorem

**Theorem:** A directed graph G is acyclic
⇔ a depth-first search of G yields no back edges.

**Proof:**

"⇒": Suppose there is a back edge (u,v). Then by definition of a back edge there would be a cycle.

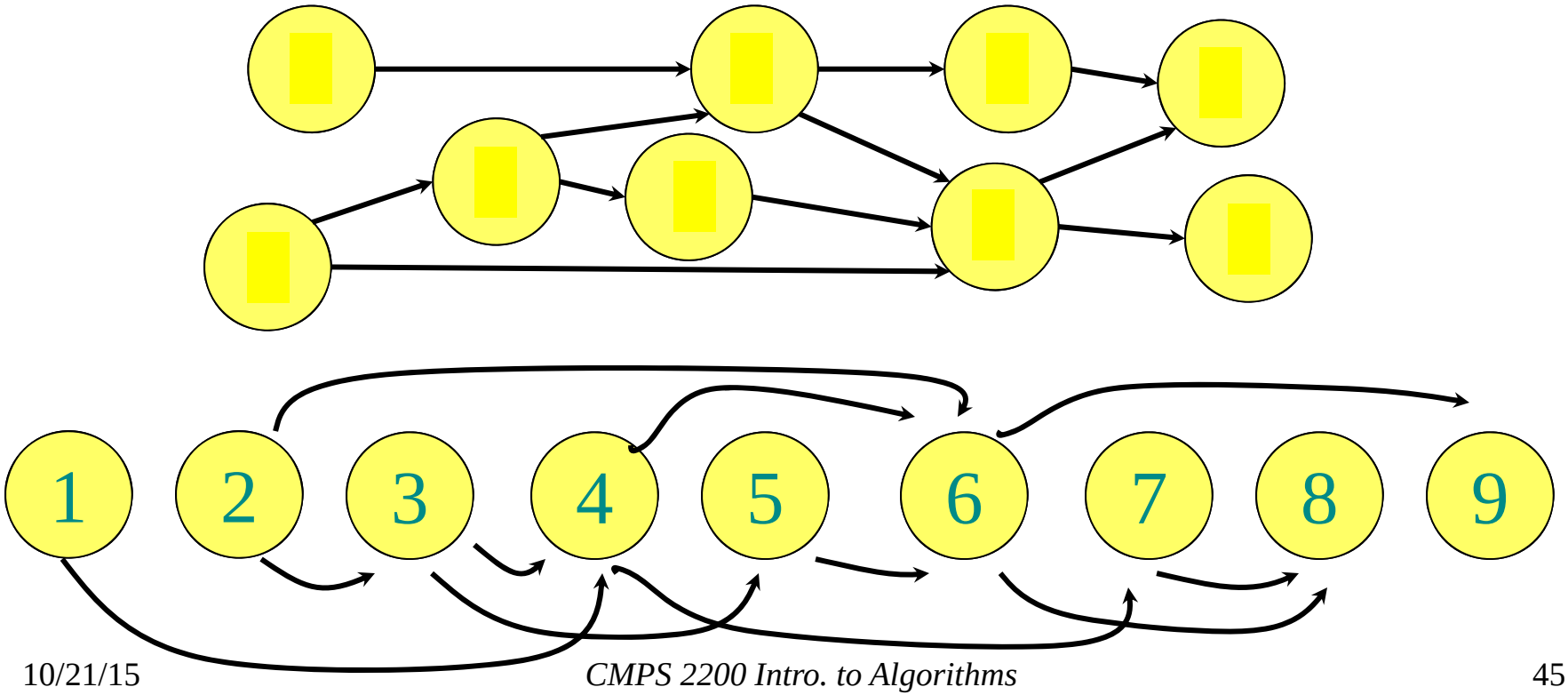
"⇐": Suppose G contains a cycle c. Let v be the first vertex to be discovered in c, and let u be the preceding vertex in c. v is an ancestor of u in the depth-first forest, hence (u,v) is a back edge.



# Topological Sort

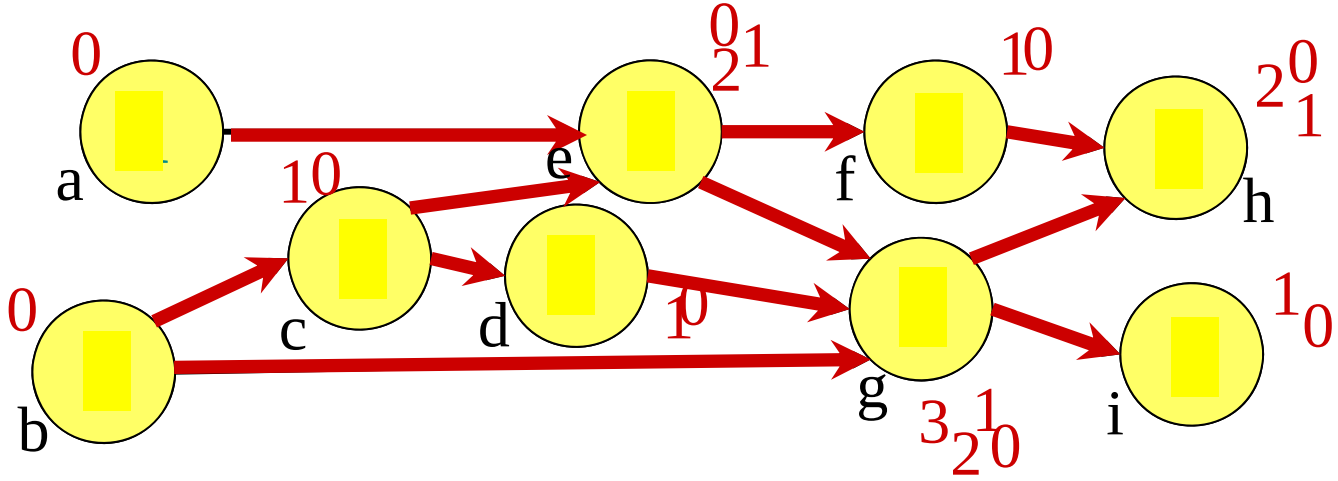
***Topologically sort*** the vertices of a ***directed acyclic graph*** (***DAG***):

- Determine $f : V \to \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$.



# Topological Sort Algorithm

- Store vertices with in-degree 0 in a queue Q.
- While Q is not empty
  - Dequeue vertex v, and give it the next number
  - Decrease in-degree of all adjacent vertices by 1
  - Enqueue all vertices with in-degree 0




Q: a , b , c, e, d, f, g, i , h

# Topological Sort Runtime

**Runtime:**

- $O(|V|+|E|)$ because every edge is touched once, and every vertex is enqueued and dequeued exactly once

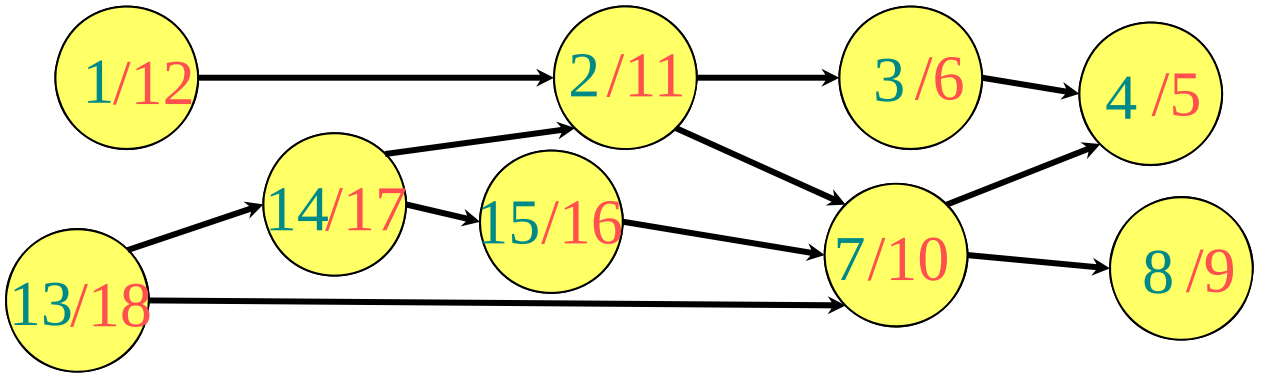
# DFS-Based Topological Sort Algorithm

- Call DFS on the directed acyclic graph $G=(V,E)$
  $\Rightarrow$ Finish time for every vertex
- Reverse the finish times (highest finish time becomes the lowest finish time,…)
  $\Rightarrow$ Valid function $f': V \rightarrow \{1, 2, \ldots, |V|\}$ such that
  $(u, v) \in E \Rightarrow f'(u) < f'(v)$

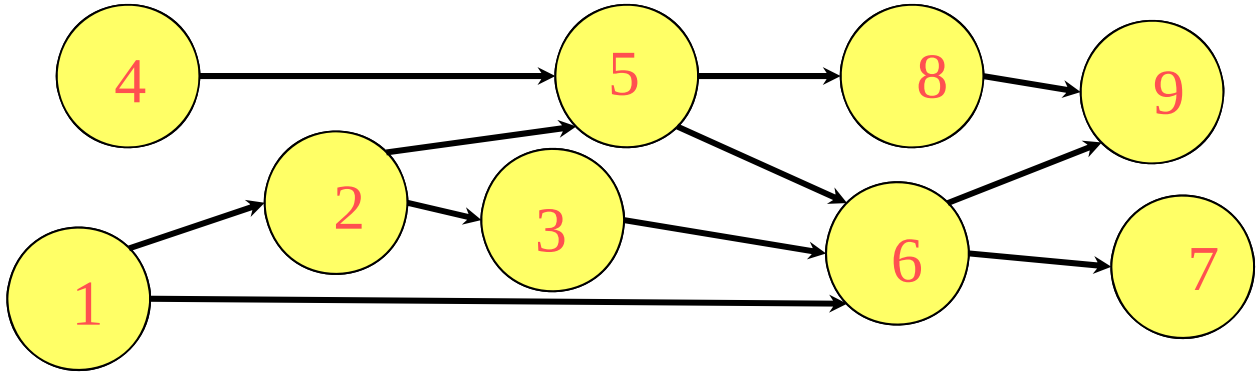
Runtime: $O(|V|+|E|)$

# DFS-Based Topological Sort

- Run DFS:




- Reverse finish times:



# DFS-Based Top. Sort Correctness

- Need to show that for any $(u, v) \in E$ holds $f(v) < f(u)$.
  (since we consider reversed finish times)
- Consider exploring edge $(u, v)$ in DFS:
  - $v$ cannot be visited and unfinished (and hence an ancestor in the depth first tree), since then $(u,v)$ would be a back edge (which by the DAG lemma cannot happen).
  - If $v$ has not been visited yet, it becomes a descendant of $u$, and hence $f(v)<f(u)$ . (tree edge)
  - If $v$ has been finished, $f(v)$ has been set, and $u$ is still being explored, hence $f(u)>f(v)$ (forward edge, cross edge) .

# Topological Sort Runtime

**Runtime:**

- $O(|V|+|E|)$ because every edge is touched once, and every vertex is enqueued and dequeued exactly once

- DFS-based algorithm: $O(|V| + |E|)$