CMPS 2200 – Fall 2014

Dynamic Programming

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Slides courtesy of Charles Leiserson with changes and additions by Carola Wenk

Dynamic programming

- Algorithm design technique
- A technique for solving problems that have
    1. an optimal substructure property (recursion)
    2. overlapping subproblems
- **Idea:** Do not repeatedly solve the same subproblems, but solve them only once and store the solutions in a **dynamic programming table**

Example: Fibonacci numbers

- $F(0)=0$; $F(1)=1$; $F(n)=F(n-1)+F(n-2)$ for $n \geq 2$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Dynamic-programming hallmark #1

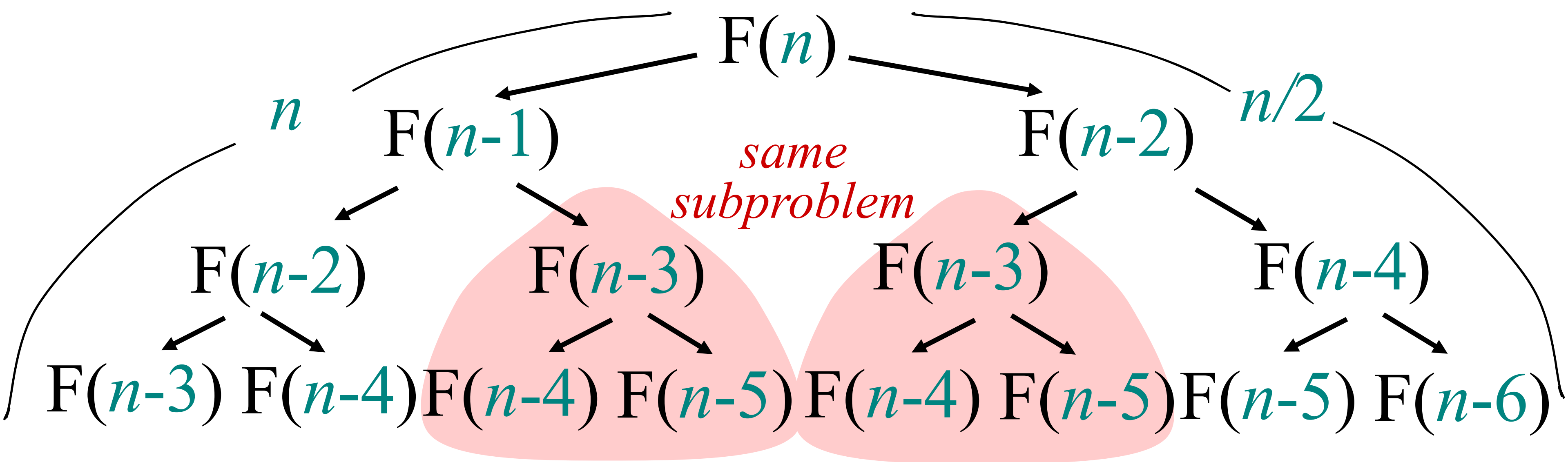
Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

⇒ ***Recursion***

Example: Fibonacci numbers

- F(0)=0; F(1)=1; F(n)=F(n-1)+F(n-2) for $n \geq 2$
- Implement this recursion directly:



- Runtime is exponential: $2^{n/2} \leq T(n) \leq 2^n$
- But we are repeatedly solving the same subproblems

Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct Fibonacci subproblems is only n.

Dynamic-programming

There are two variants of dynamic programming:

1. Bottom-up dynamic programming (often referred to as "dynamic programming")

2. Memoization

Bottom-up dynamic-programming algorithm

- Store 1D DP-table and fill bottom-up:

F:

0	1	1	2	3	5	8				

fibBottomUpDP(n)
  F[0] ← 0
  F[1] ← 1
  for ($i \leftarrow 2,\ i \leq n,\ i{+}{+}$)
    F[i] ← F[i-1]+F[i-2]
  return F[n]

- Time $= \Theta(n)$, space $= \Theta(n)$

Memoization algorithm

Memoization: Use recursive algorithm. After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
fibMemoization(n)
    for all i: F[i] = null
    fibMemoizationRec(n,F)
    return F[n]

fibMemoizationRec(n,F)
    if (F[n]= null)
            if (n=0) F[n] ← 0
            if (n=1) F[n] ← 1
            F[n] ← fibMemoizationRec(n-1,F)
                   + fibMemoizationRec(n-2,F)
    return F[n]
```

- Time = $\Theta(n)$, space = $\Theta(n)$

Longest Common Subsequence

Example: *Longest Common Subsequence (LCS)*

- Given two sequences $x[1 \,.\,.\, m]$ and $y[1 \,.\,.\, n]$, find a longest subsequence common to them both.

"a" *not* "the"

x: A B C B D A B

y: B D C A B A

BCBA = LCS(x, y)

functional notation, but not a function

Brute-force LCS algorithm

Check every subsequence of $x[1 \mathinner{.\,.} m]$ to see if it is also a subsequence of $y[1 \mathinner{.\,.} n]$.

Analysis

- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).
- Hence, the runtime would be exponential !

Towards a better algorithm

Two-Step Approach:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider ***prefixes*** of x and y.

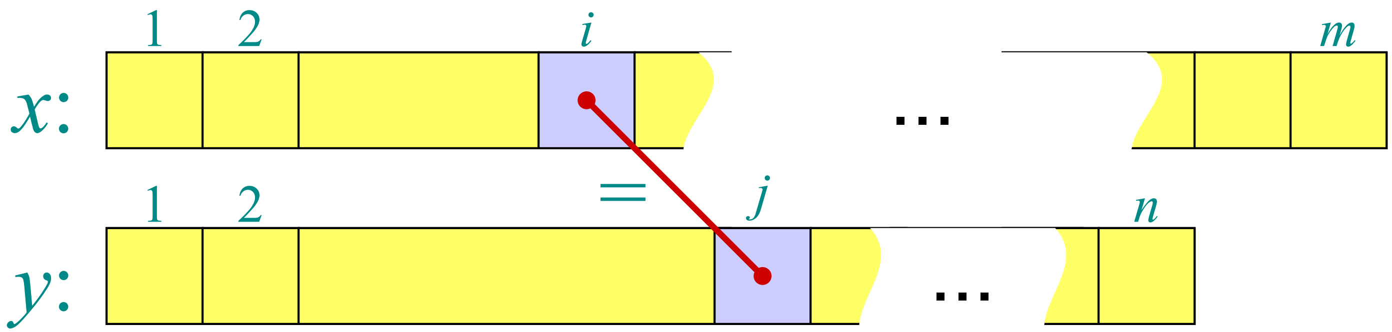
- Define $c[i, j] = |\text{LCS}(x[1 \,.\,.\, i], y[1 \,.\,.\, j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.

Recursive formulation

Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

Proof. Case $x[i] = y[j]$:



Let $z[1 \,.\,.\, k] = \text{LCS}(x[1 \,.\,.\, i], y[1 \,.\,.\, j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \,.\,.\, k-1]$ is CS of $x[1 \,.\,.\, i-1]$ and $y[1 \,.\,.\, j-1]$.

Proof (continued)

Claim: $z[1\,..\,k-1] = \mathrm{LCS}(x[1\,..\,i-1], y[1\,..\,j-1])$.

Suppose w is a longer CS of $x[1\,..\,i-1]$ and $y[1\,..\,j-1]$, that is, $|w| > k-1$. Then, ***cut and paste***: $w \,||\, z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1\,..\,i]$ and $y[1\,..\,j]$ with $\big|w \,||\, z[k]\big| > k$. Contradiction, proving the claim.

Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

Other cases are similar. $\square$

Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

→ ***Recursion***

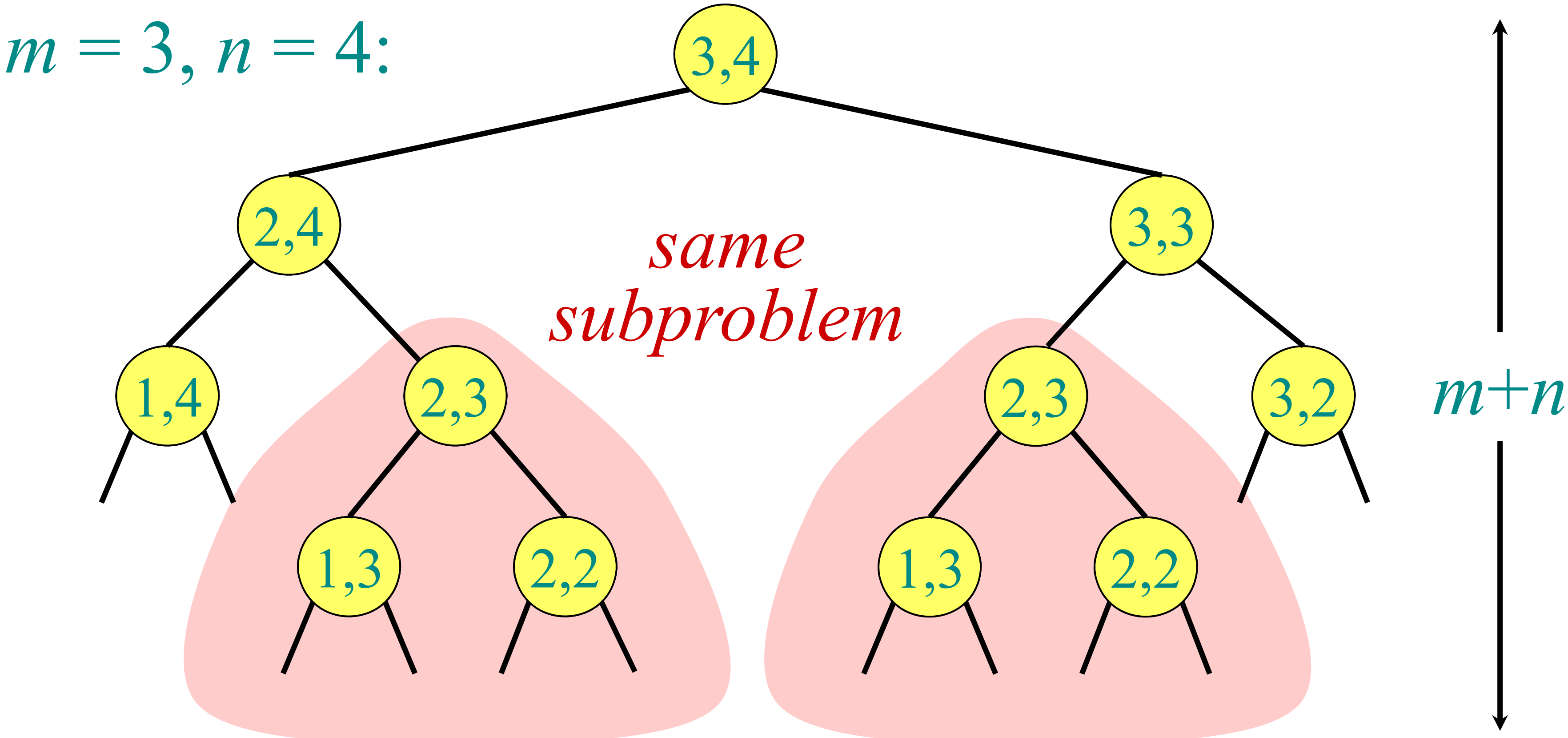
If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

Recursive algorithm for LCS

LCS(x, y, i, j)
    if $(i{=}0 \text{ or } j{=}0)$
        $c[i, j] \leftarrow 0$
    else if $x[i] = y[j]$
        $c[i, j] \leftarrow$ LCS$(x, y, i{-}1, j{-}1) + 1$
    else $c[i, j] \leftarrow \max\{LCS(x, y, i{-}1, j)$,
                        LCS$(x, y, i, j{-}1)\}$
    return $c[i, j]$

Worst-case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Recursion tree



Height $= m + n \Rightarrow$ work potentially exponential, but we're solving subproblems already solved!

Dynamic-programming hallmark #2

Overlapping subproblems
A recursive solution contains a "small" number of distinct subproblems repeated many times.

The distinct LCS subproblems are all the pairs (i,j). The number of such pairs for two strings of lengths m and n is only mn.

Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

LCS(x, y, i, j)
    if $c[i, j] = \text{NIL}$
        if $(i=0 \text{ or } j=0)$
            $c[i, j] \leftarrow 0$
        else if $x[i] = y[j]$
            $c[i, j] \leftarrow$ LCS$(x, y, i-1, j-1) + 1$
        else $c[i, j] \leftarrow \max\{LCS(x, y, i-1, j)$,
                            LCS$(x, y, i, j-1)\}$
    return $c[i, j]$

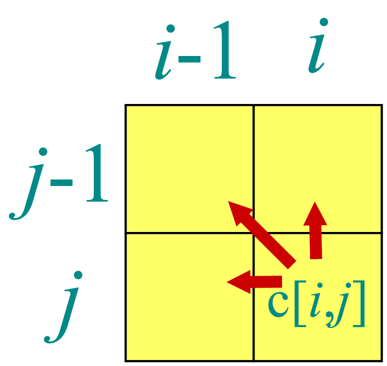
same as before

Space = time $= \Theta(mn)$; constant work per table entry.

Recursive formulation

$$c[i,j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

c:



Bottom-up dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

$y \downarrow$ \ $x \rightarrow$		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Bottom-up dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by back-tracking.

Space = $\Theta(mn)$.

Exercise: $O(\min\{m, n\})$.

$y \downarrow$ \ $x \rightarrow$		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4