

CMPS 2200 – Fall 2014

Single Source Shortest Paths

Carola Wenk

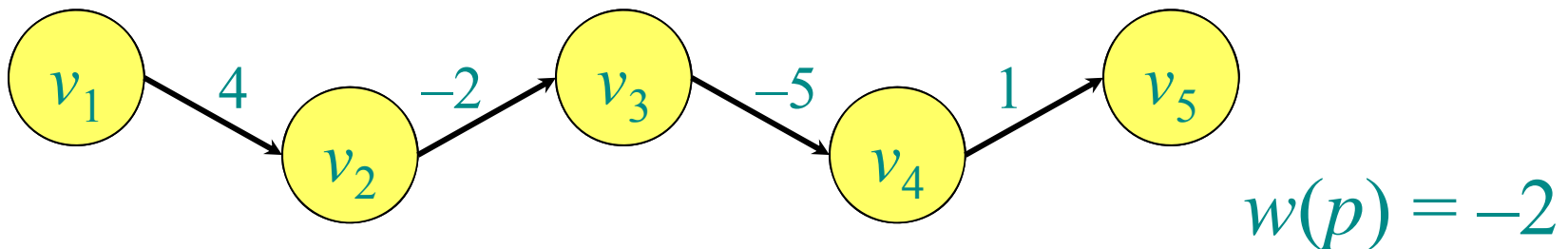
Slides courtesy of Charles Leiserson with changes
and additions by Carola Wenk

Paths in graphs

Consider a digraph $G = (V, E)$ with an edge-weight function $w : E \rightarrow \mathbb{R}$. The **weight** of path $p = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

Example:



Shortest paths

A *shortest path* from u to v is a path of minimum weight from u to v .

The *shortest-path weight* from u to v is defined as

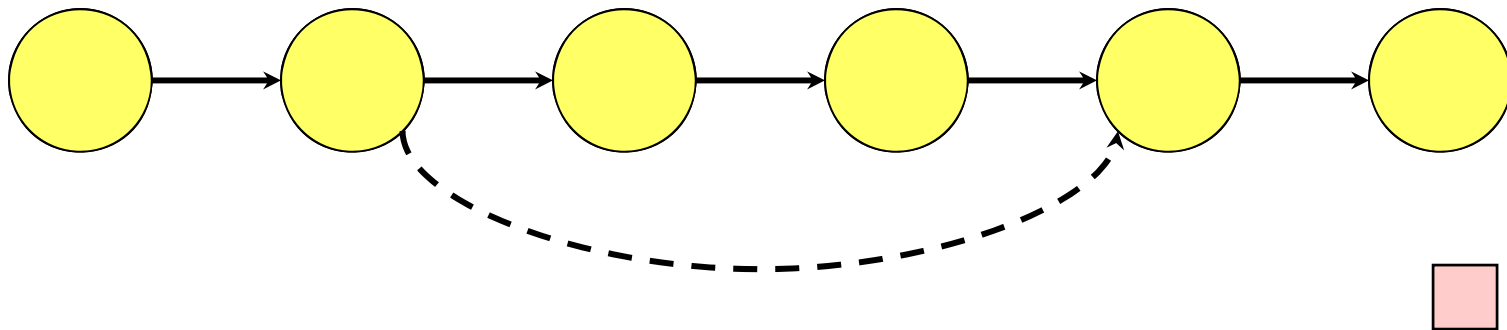
$$\delta(u, v) = \min \{w(p) : p \text{ is a path from } u \text{ to } v\}.$$

Note: $\delta(u, v) = \infty$ if no path from u to v exists.

Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:

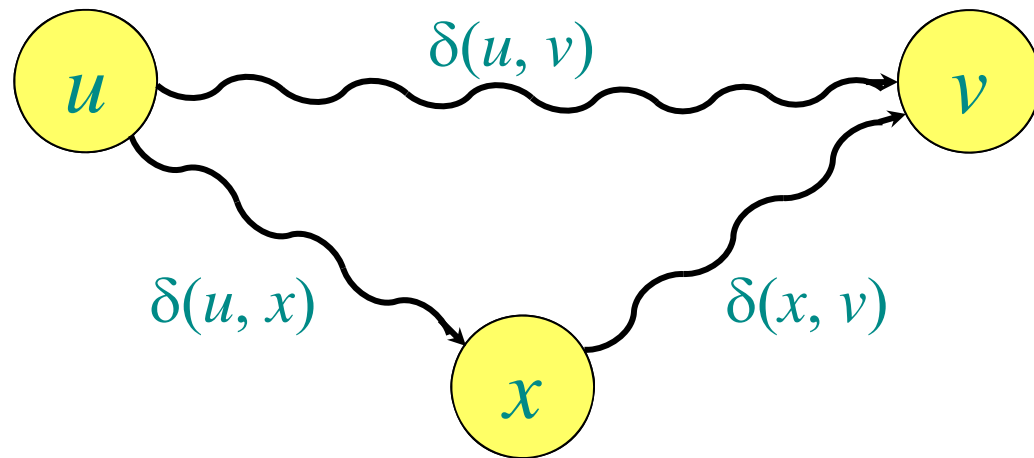


Triangle inequality

Theorem. For all $u, v, x \in V$, we have
$$\delta(u, v) \leq \delta(u, x) + \delta(x, v).$$

Proof.

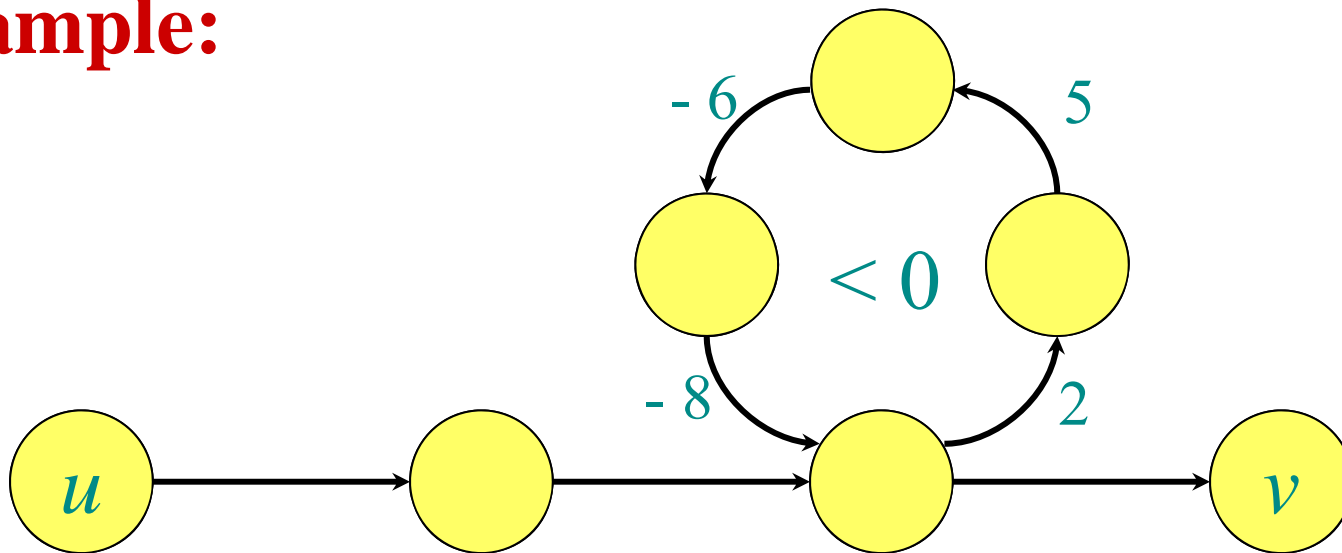
- $\delta(u, v)$ minimizes over **all** paths from u to v
- Concatenating two shortest paths from u to x and from x to v yields **one** specific path from u to v



Well-definedness of shortest paths

If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Single-source shortest paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

Assumption:

All edge weights $w(u, v)$ are *non-negative*.

It follows that all shortest-path weights must exist.

IDEA: Greedy.

1. Maintain a set S of vertices whose shortest-path weights from s are known, i.e., $d[v] = \delta(s, v)$
2. At each step add to S the vertex $u \in V - S$ whose distance estimate $d[u]$ from s is minimal.
3. Update the distance estimates $d[v]$ of vertices v adjacent to u .

Dijkstra's algorithm

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

do $d[v] \leftarrow \infty$

$S \leftarrow \emptyset$ ▷ Vertices for which $d[v]=d(s,v)$

$Q \leftarrow V$ ▷ Q is a priority queue maintaining $V - S$
sorted by d -values $d[v]$

while $Q \neq \emptyset$ **do**

$u \leftarrow \text{EXTRACT-MIN}(Q)$

$S \leftarrow S \cup \{u\}$

for each $v \in \text{Adj}[u]$ **do**

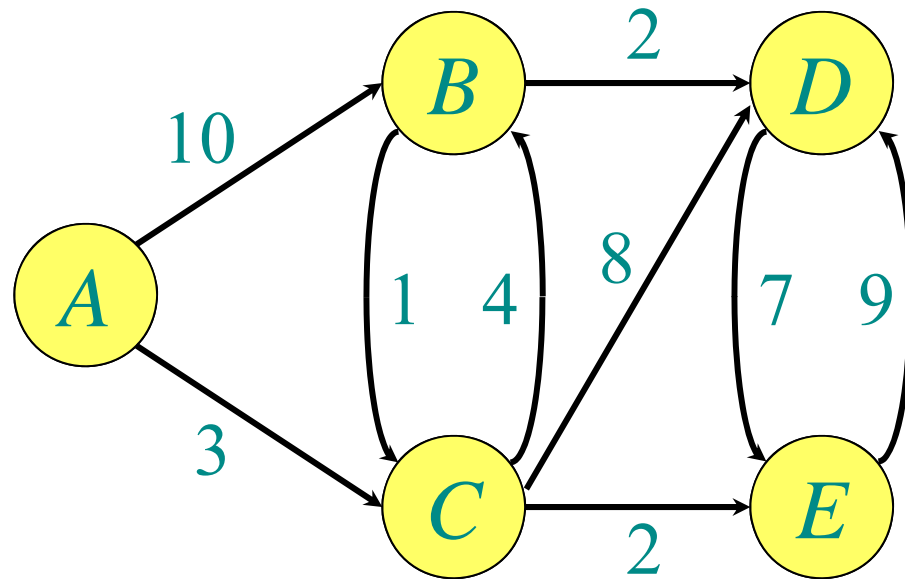
if $d[v] > d[u] + w(u, v)$ **then**
 $d[v] \leftarrow d[u] + w(u, v)$

relaxation step

implicit DECREASE-KEY in Q

Example of Dijkstra's algorithm

Graph with nonnegative edge weights:



```
while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
```

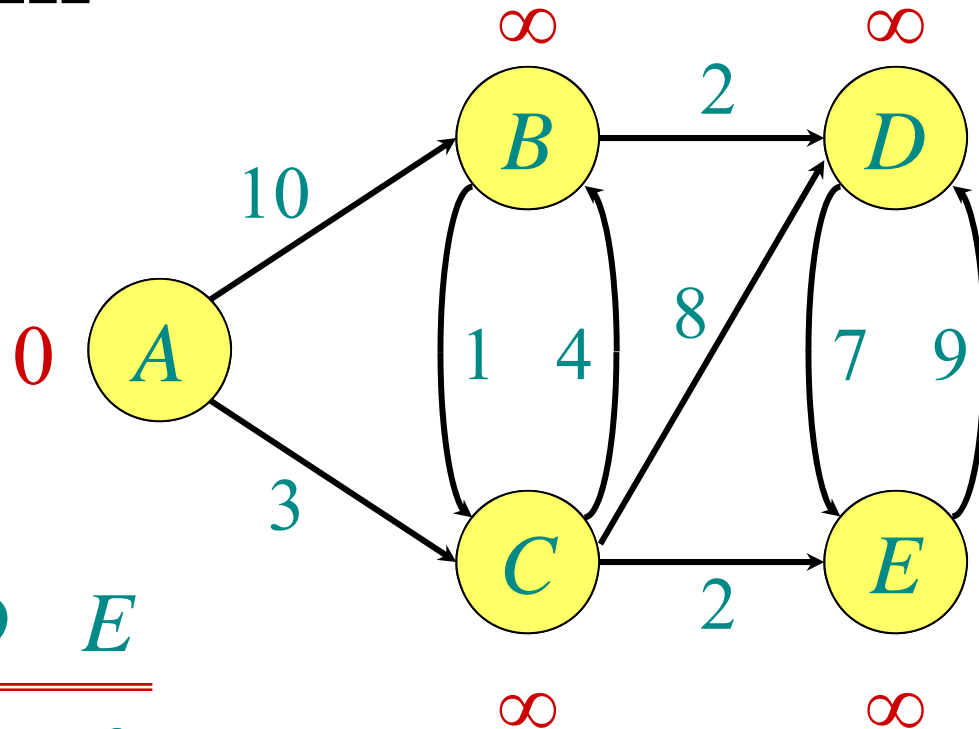
Example of Dijkstra's algorithm

Initialize:

$S: \{\}$

$Q:$

<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
0	∞	∞	∞	∞



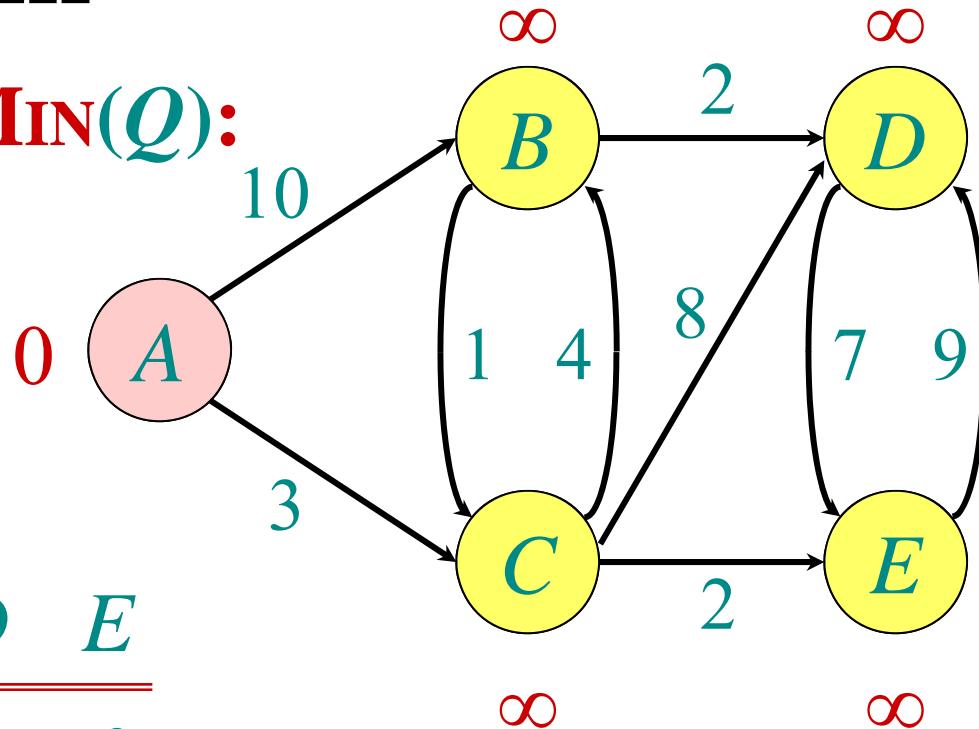
```
while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
```

Example of Dijkstra's algorithm

“A” ← **EXTRACT-MIN**(Q):

S: { A }

Q:	A	B	C	D	E
	0	∞	∞	∞	∞



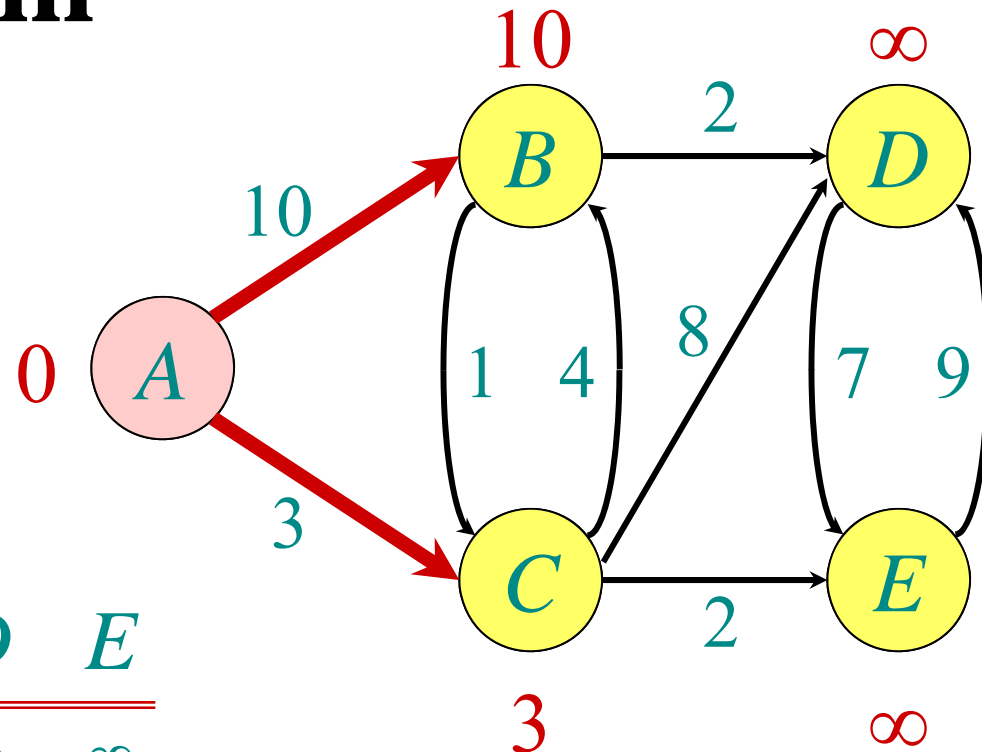
```

while Q ≠ ∅ do
  u ← EXTRACT-MIN(Q)
  S ← S ∪ {u}
  for each v ∈ Adj[u] do
    if d[v] > d[u] + w(u, v) then
      d[v] ← d[u] + w(u, v)
  
```

Example of Dijkstra's algorithm

Relax all edges leaving A :

$S: \{A\}$



$Q:$	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	—	—

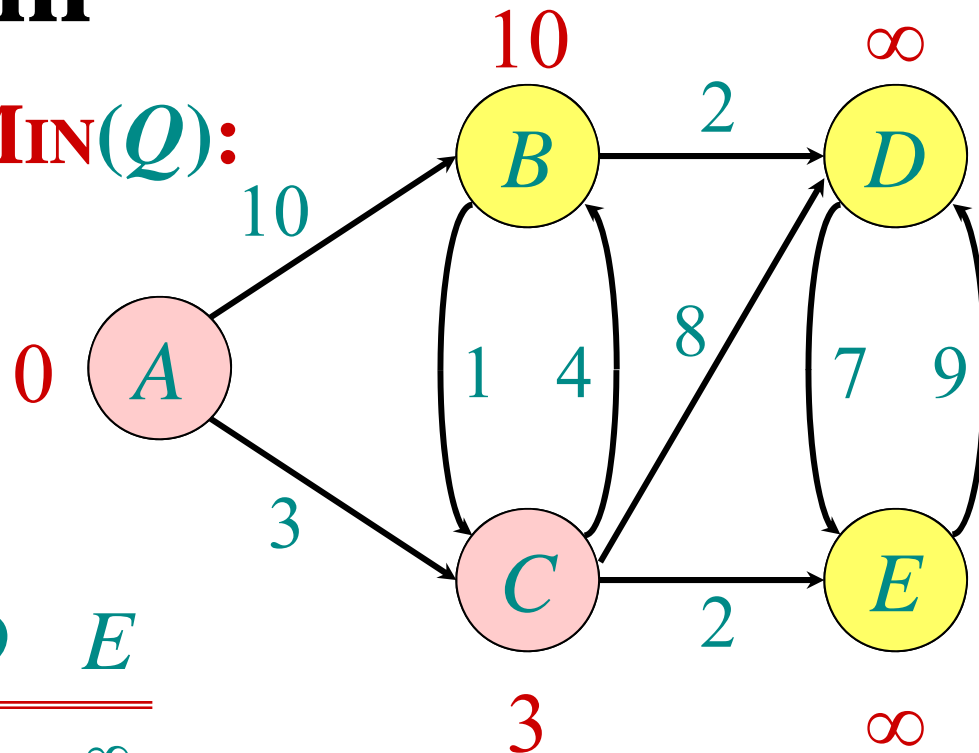
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
  
```

Example of Dijkstra's algorithm

“C” ← **EXTRACT-MIN(Q)**:

$S: \{A, C\}$



$Q:$	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	—	—

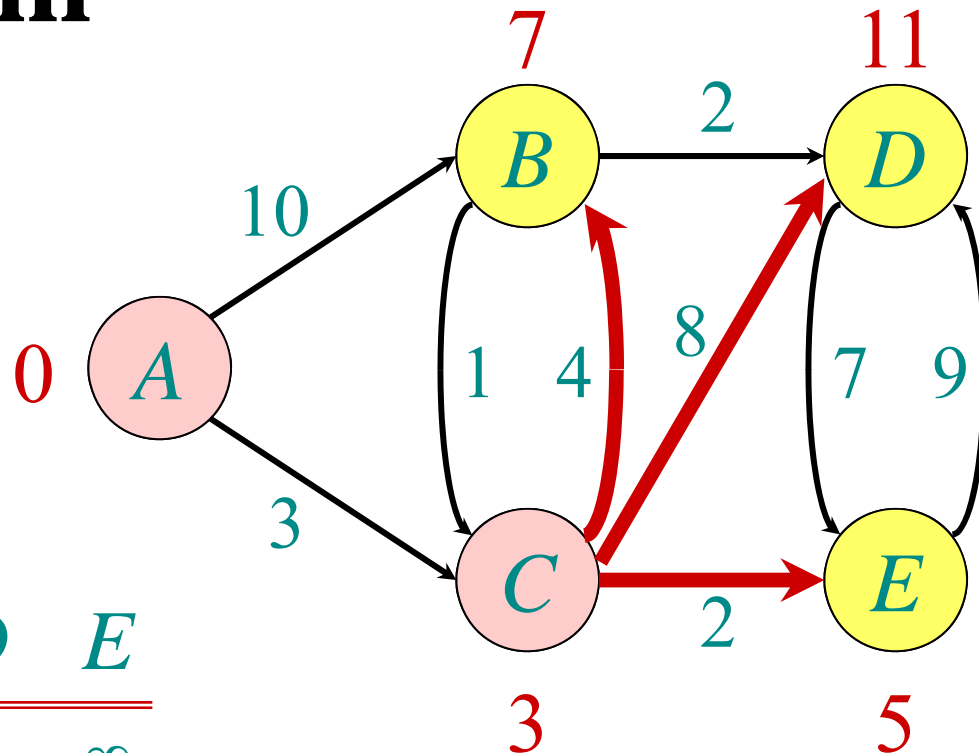
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
  
```

Example of Dijkstra's algorithm

Relax all edges leaving C :

$S: \{A, C\}$



$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	-	-
	7		11	5

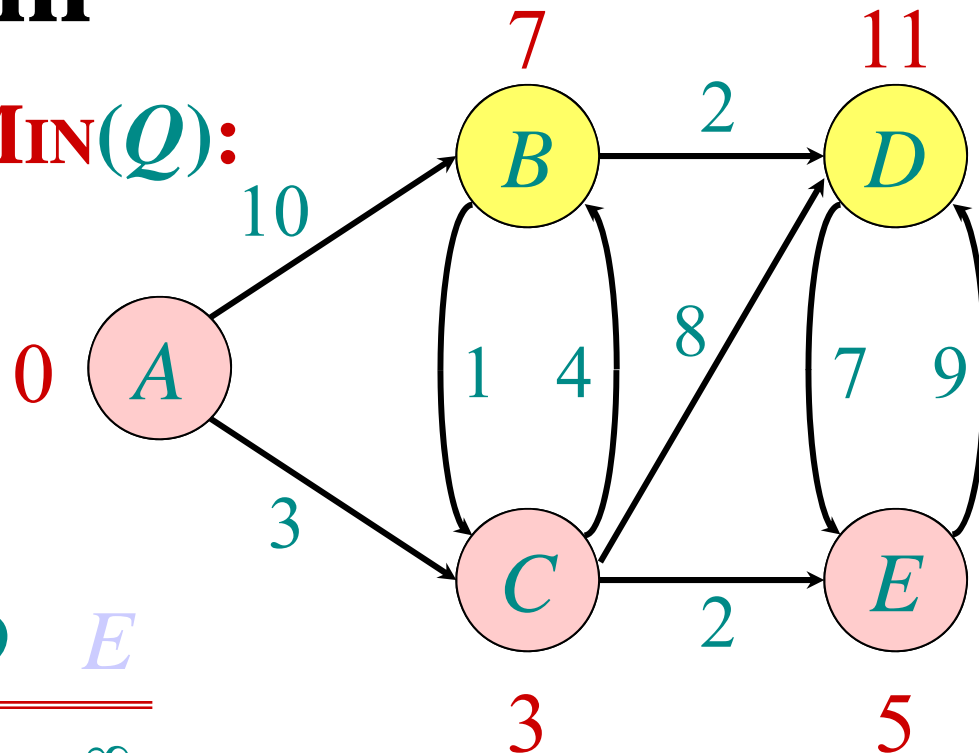
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
  
```

Example of Dijkstra's algorithm

“E” ← **EXTRACT-MIN(Q)**:

$S: \{A, C, E\}$



Q:

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	—	—
	7		11	5

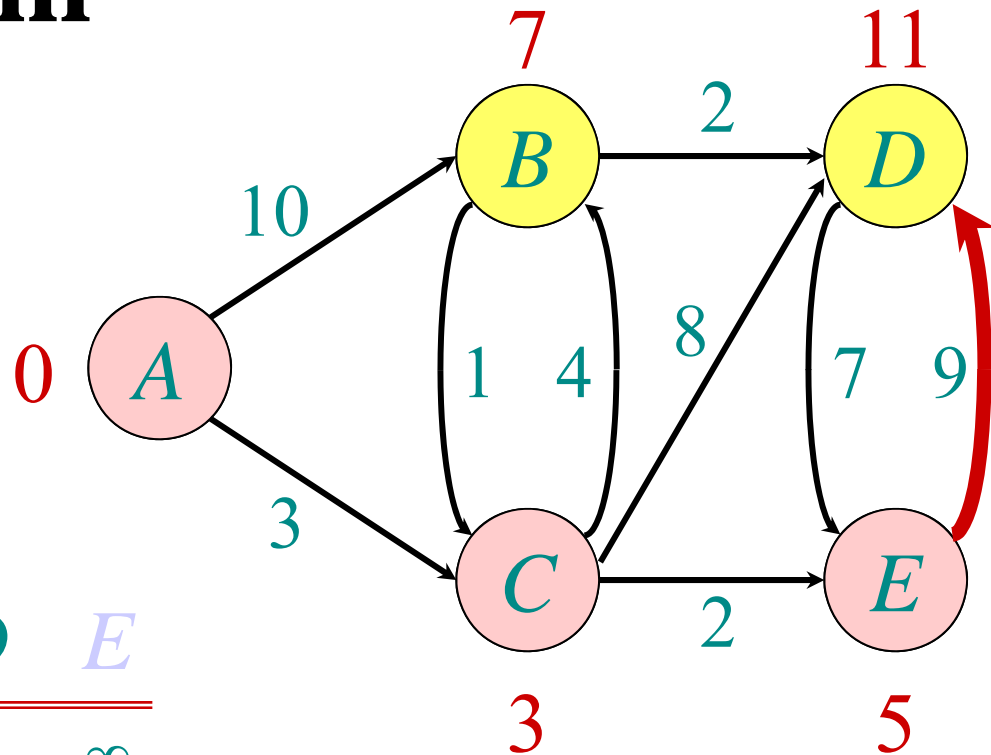
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
  
```

Example of Dijkstra's algorithm

Relax all edges leaving E :

$S: \{A, C, E\}$



$Q:$	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

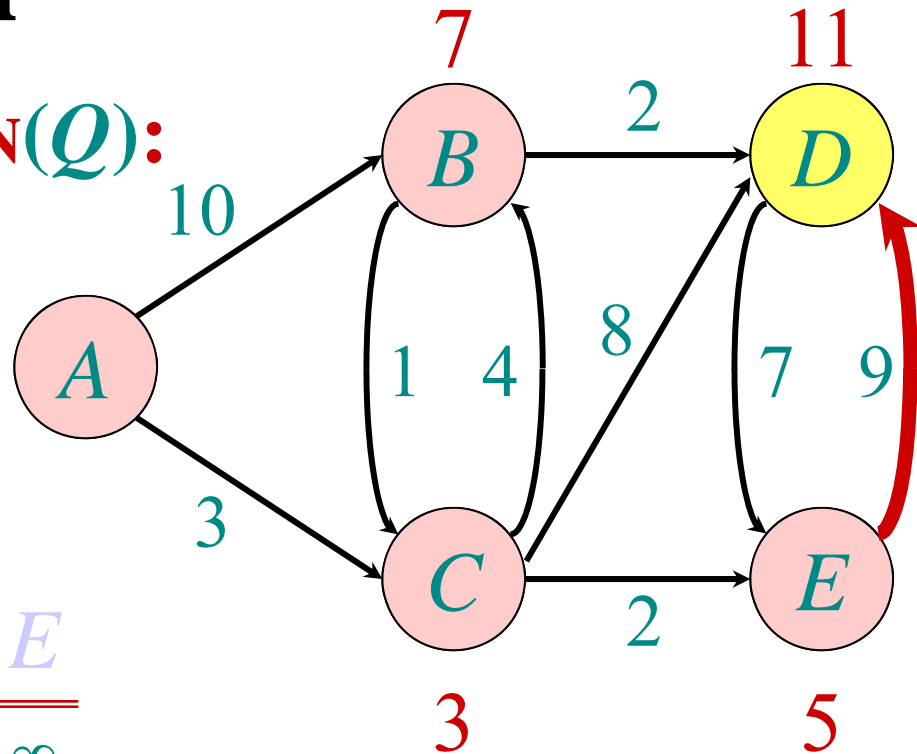
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
    
```


Example of Dijkstra's algorithm

“B” ← **EXTRACT-MIN(Q)**:

$S: \{A, C, E, B\}$ 0



$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	

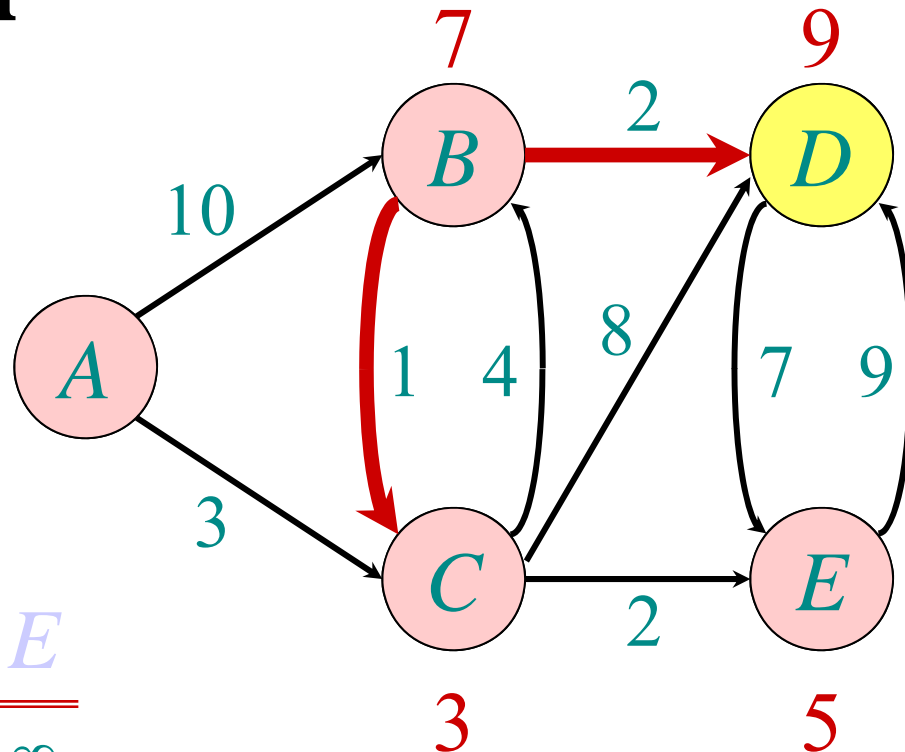
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
  
```

Example of Dijkstra's algorithm

Relax all edges leaving B :

$S: \{A, C, E, B\}$ 0



$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	

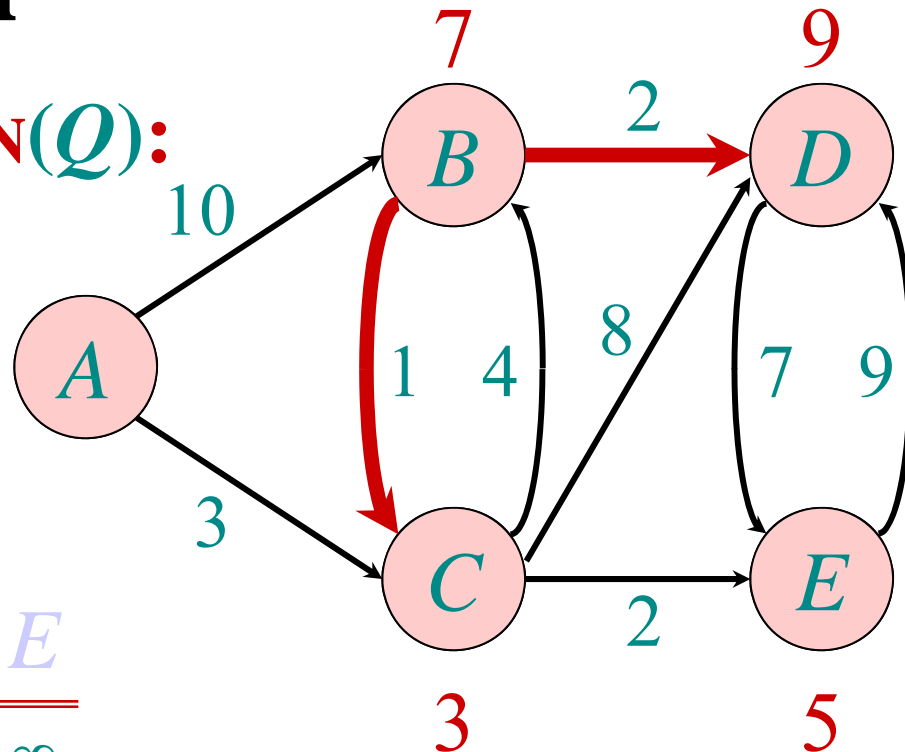
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
  
```

Example of Dijkstra's algorithm

“D” ← **EXTRACT-MIN(Q)**:

$S: \{A, C, E, B, D\}$



$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	
			9	


```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
  
```

Analysis of Dijkstra

$|V|$ times { $degree(u)$ times {

```
while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
```



Handshaking Lemma $\Rightarrow \Theta(|E|)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(|V|) \cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|) \cdot T_{\text{DECREASE-KEY}}$$

Analysis of Dijkstra (continued)

$$\text{Time} = \Theta(|V|) \cdot T_{\text{EXTRACT-MIN}} + \Theta(|E|) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V ^2)$
binary heap	$O(\log V)$	$O(\log V)$	$O(E \log V)$
Fibonacci heap	$O(\log V)$ amortized	$O(1)$ amortized	$O(E + V \log V)$ worst case

Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v]$ = weight of shortest path from s to v that uses only (besides v itself) vertices in S .

Corollary. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v] =$ weight of shortest path from s to v that uses only (besides v itself) vertices in S .

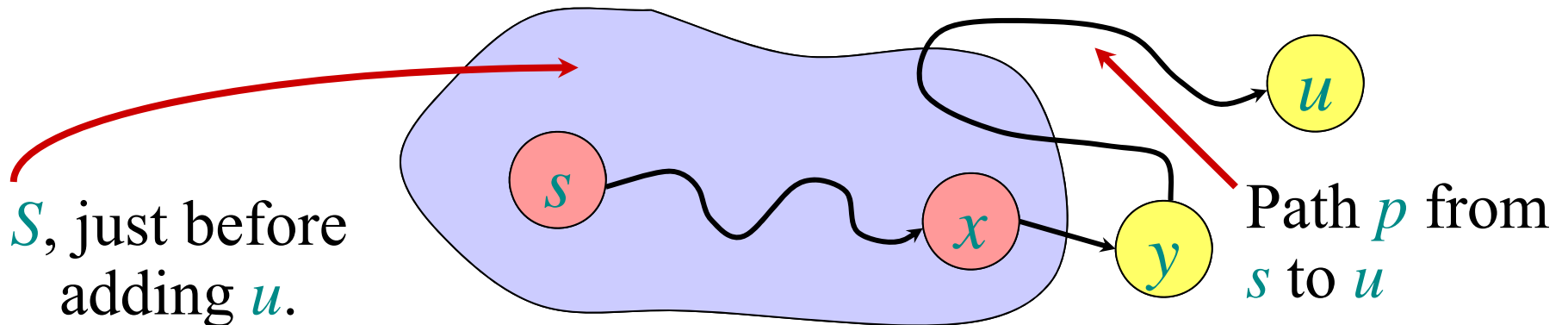
Proof. By induction.

- Base: Before the while loop, $d[s]=0$ and $d[v]=\infty$ for all $v \neq s$, so (i) and (ii) are true.
- Step: Assume (i) and (ii) are true before an iteration; now we need to show they remain true after another iteration. Let u be the vertex added to S , so $d[u] \leq d[v]$ for all other $v \notin S$.

Correctness

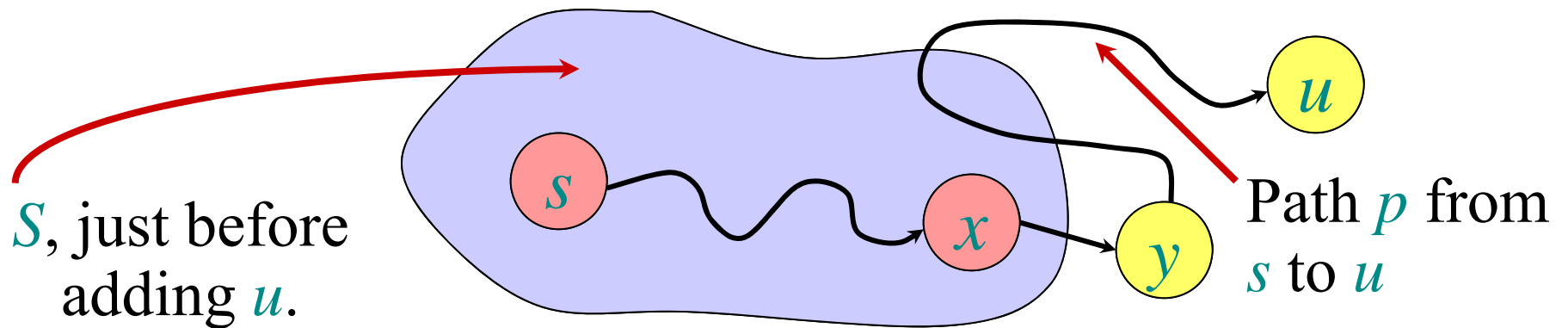
Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v] =$ weight of shortest path from s to v that uses only (besides v itself) vertices in S .

- (i) Need to show that $d[u] = \delta(s, u)$. Assume the contrary.
 \Rightarrow There is a path p from s to u with $w(p) < d[u]$. Because of (ii) that path uses vertices $\notin S$, in addition to u .
 \Rightarrow Let y be first vertex on p such that $y \notin S$.



Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v] =$ weight of shortest path from s to v that uses only (besides v itself) vertices in S .



$\Rightarrow d[y] \leq w(p) < d[u]$. Contradiction to the choice of u .

weights are nonnegative

assumption about path

Correctness

Theorem. (i) For all $v \in S$: $d[v] = \delta(s, v)$
(ii) For all $v \notin S$: $d[v] =$ weight of shortest path from s to v that uses only (besides v itself) vertices in S .

- (ii) Let $v \notin S$. Let p be a shortest path from s to v that uses only (besides v itself) vertices in S .
 - p does not contain u : (ii) true by inductive hypothesis
 - p contains u : p consists of vertices in $S \setminus \{u\}$ and ends with an edge from u to v .
 $\Rightarrow w(p) = d[u] + w(u, v)$, which is the value of $d[v]$ after adding u . So (ii) is true.

Unweighted graphs

Suppose $w(u, v) = 1$ for all $(u, v) \in E$. Can the code for Dijkstra be improved?

- Use a simple FIFO queue instead of a priority queue.

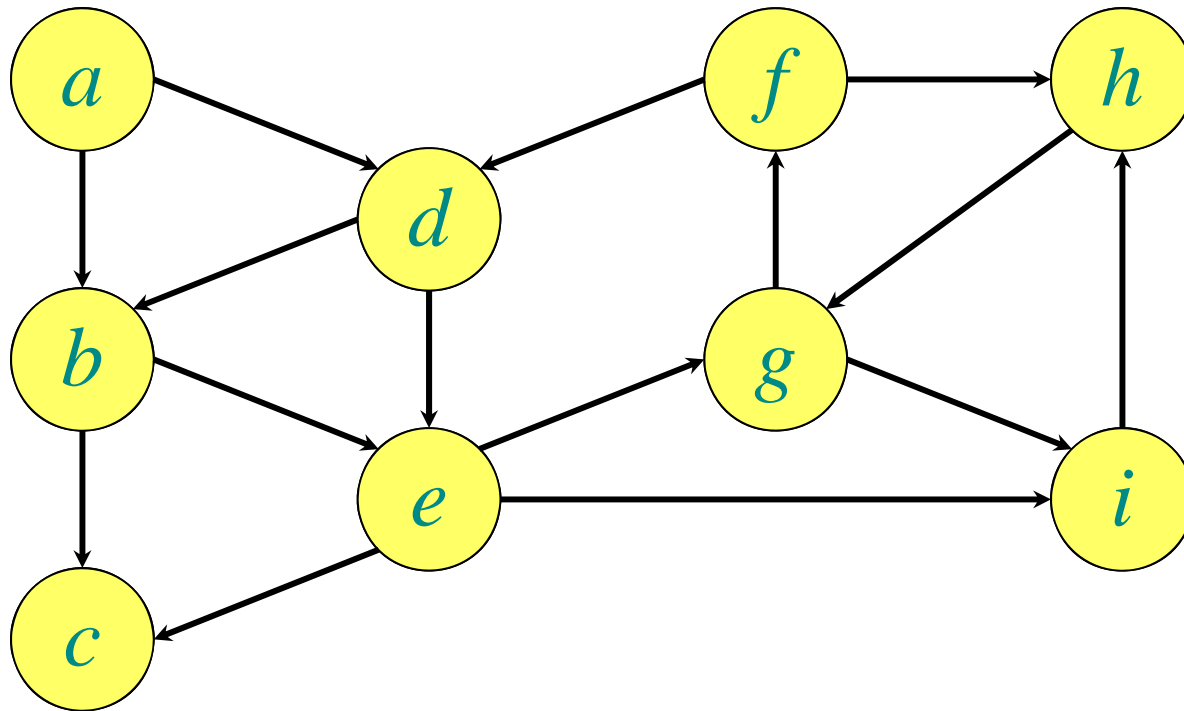
- ***Breadth-first search***

```
while  $Q \neq \emptyset$ 
do  $u \leftarrow \text{DEQUEUE}(Q)$ 
  for each  $v \in \text{Adj}[u]$ 
  do if  $d[v] = \infty$ 
    then  $d[v] \leftarrow d[u] + 1$ 
      ENQUEUE( $Q, v$ )
```

```
while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
```

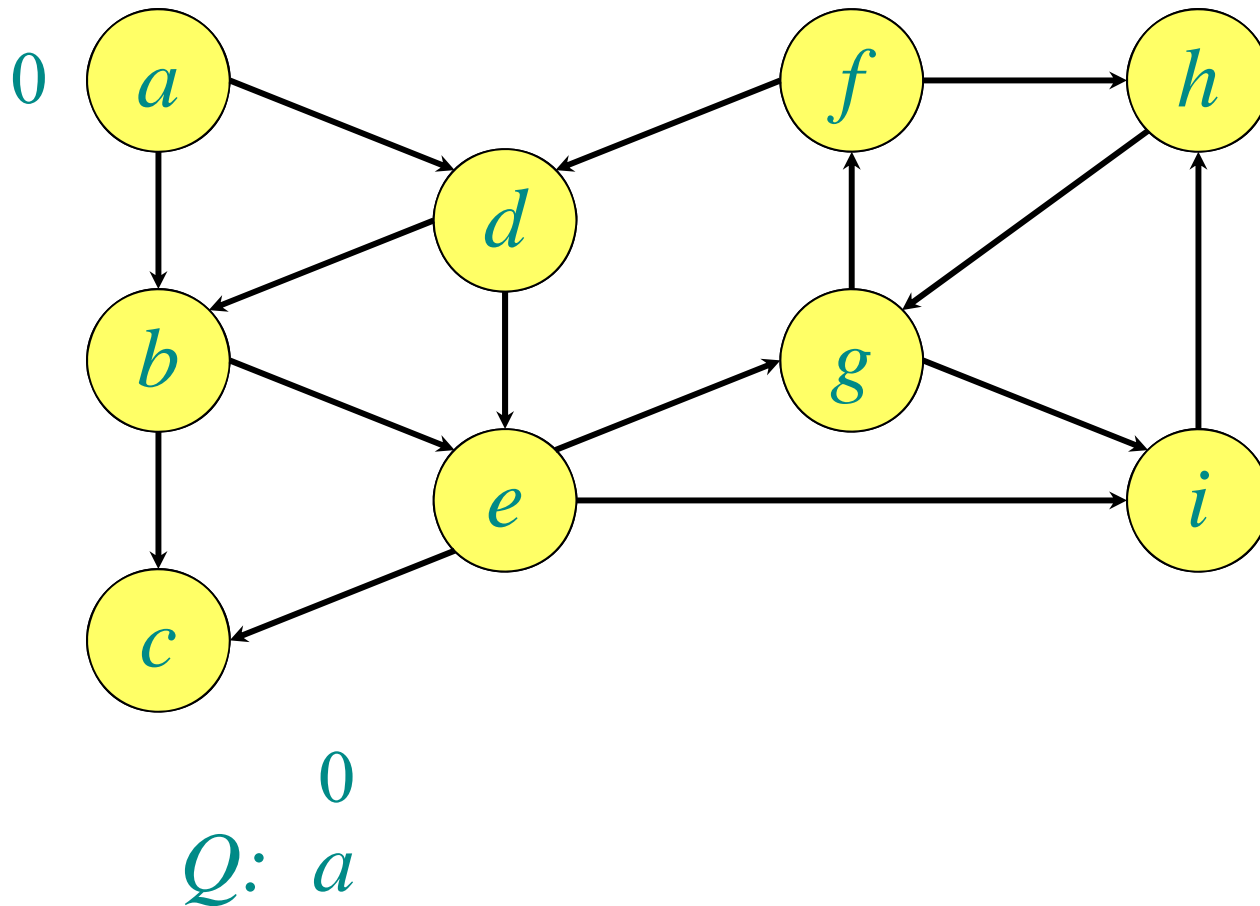
Analysis: Time = $O(|V| + |E|)$.

Example of breadth-first search

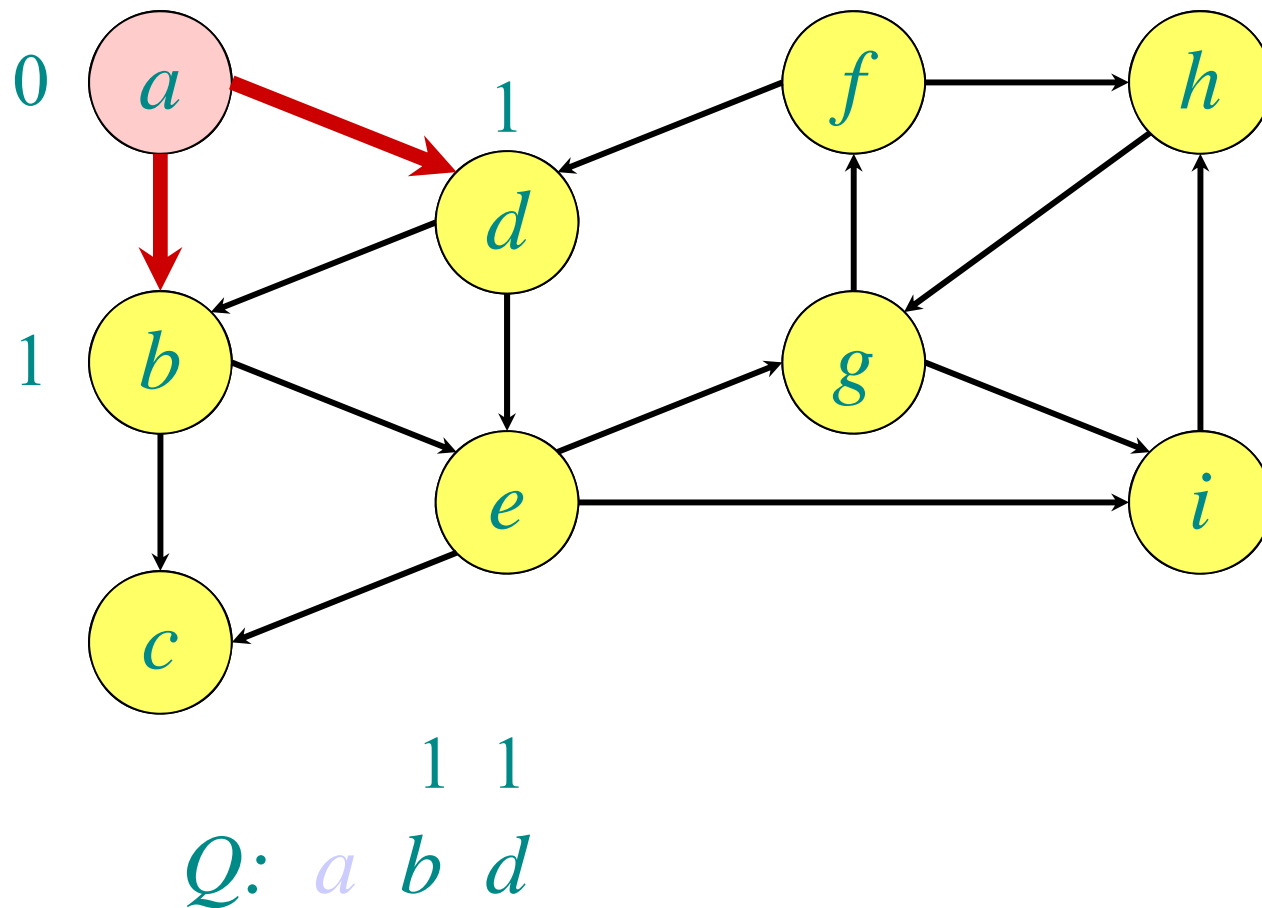


Q:

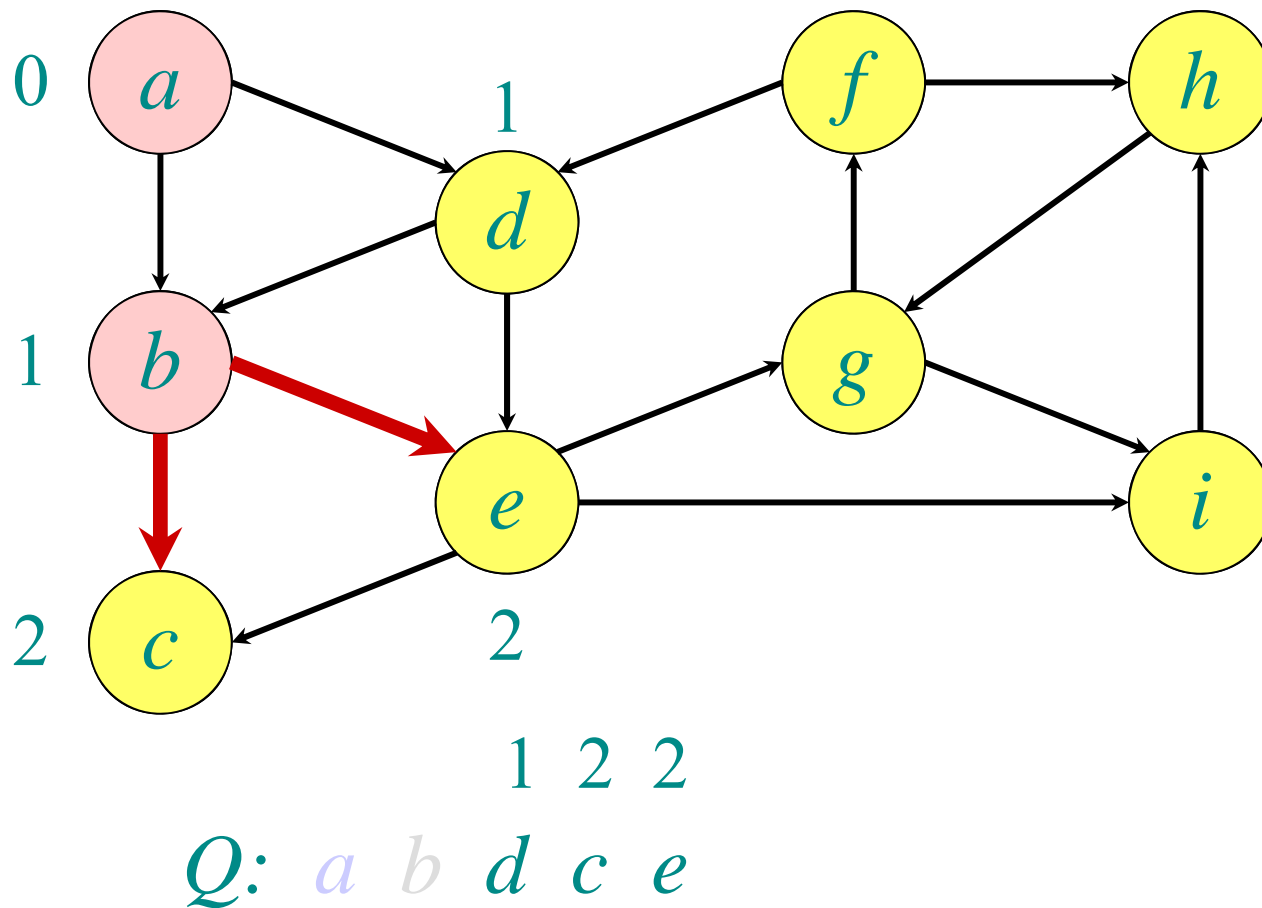
Example of breadth-first search



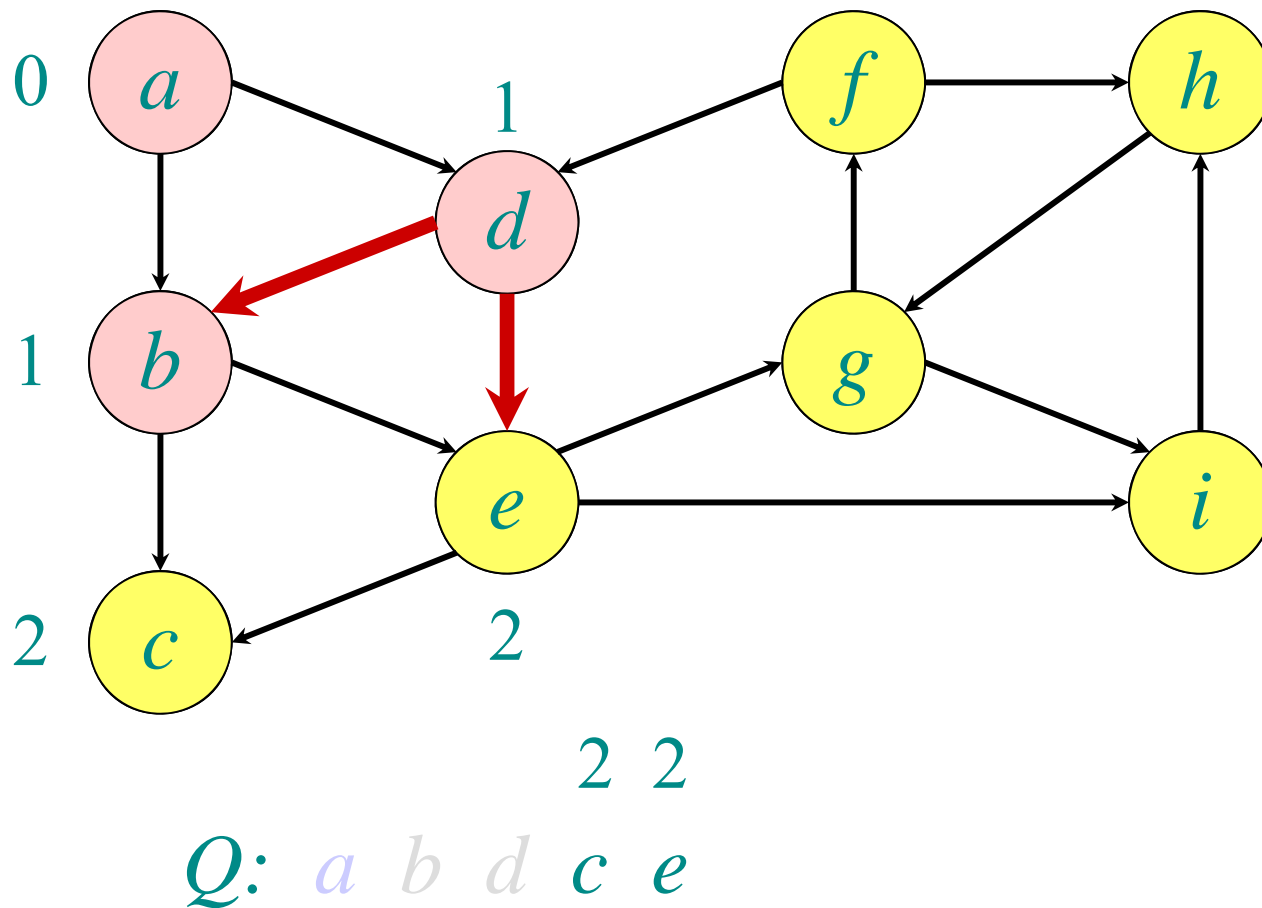
Example of breadth-first search



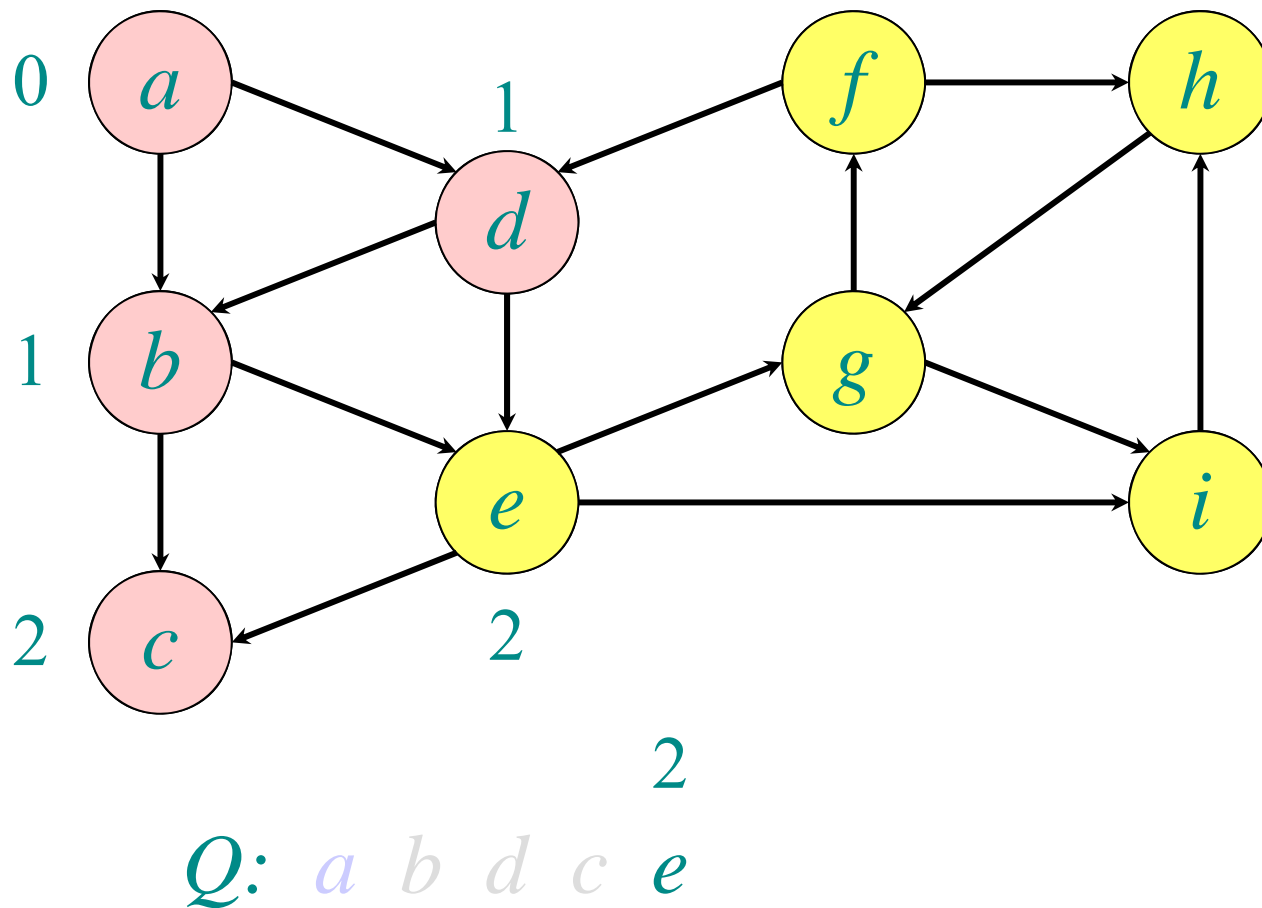
Example of breadth-first search



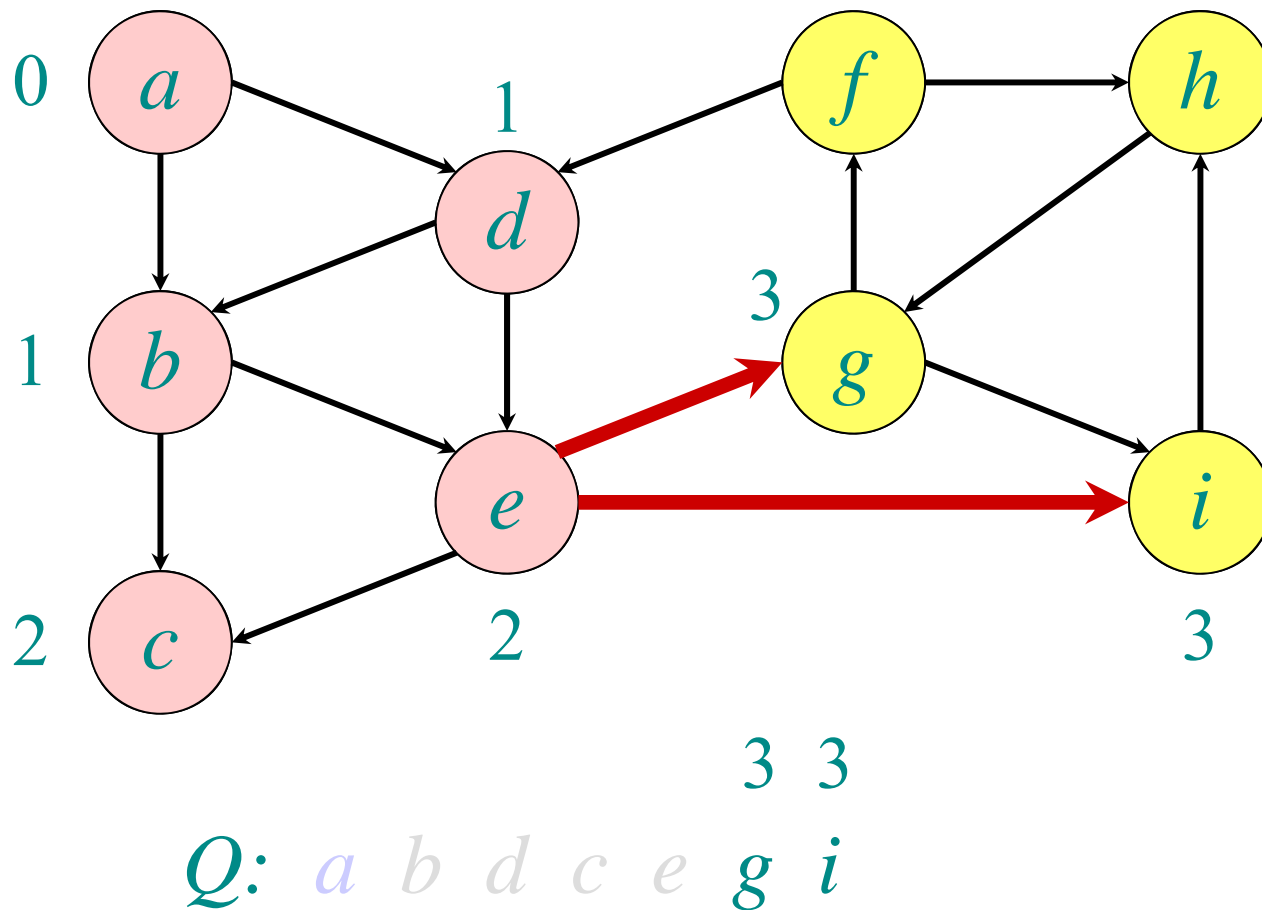
Example of breadth-first search



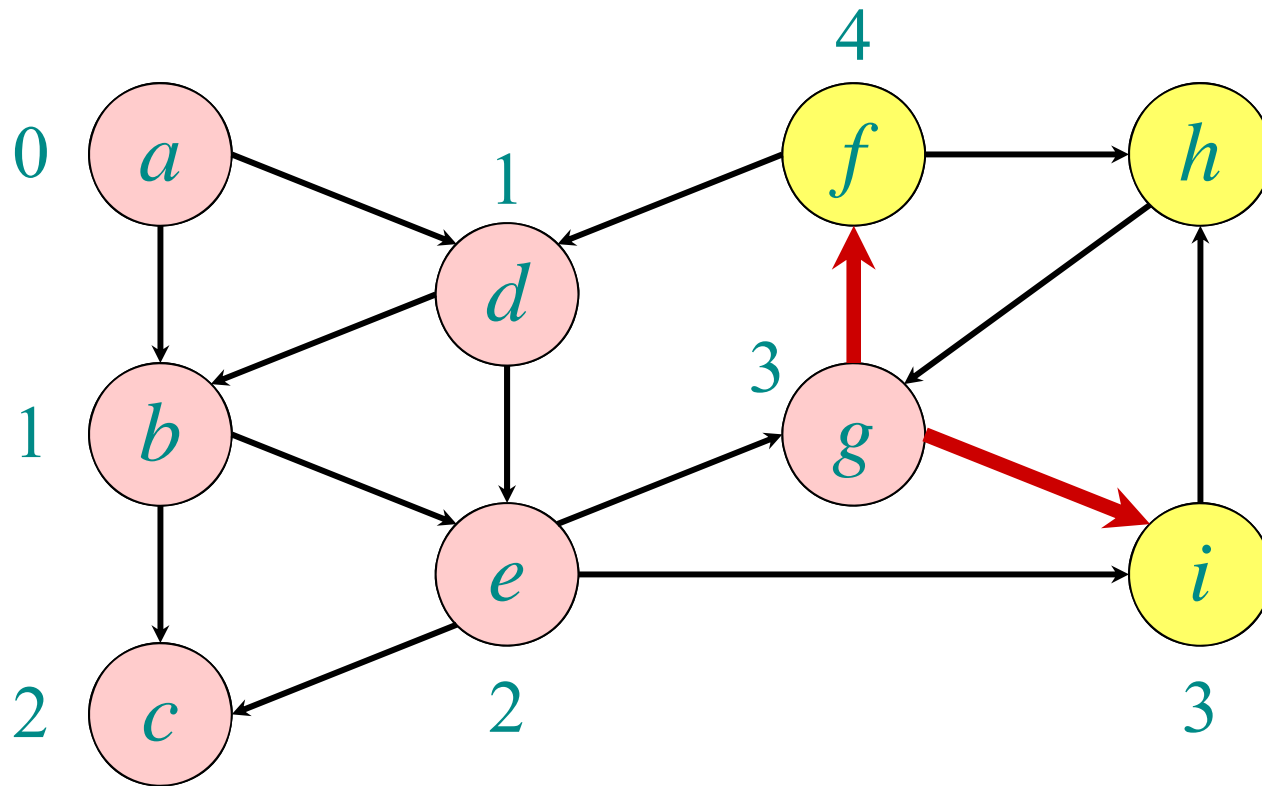
Example of breadth-first search



Example of breadth-first search

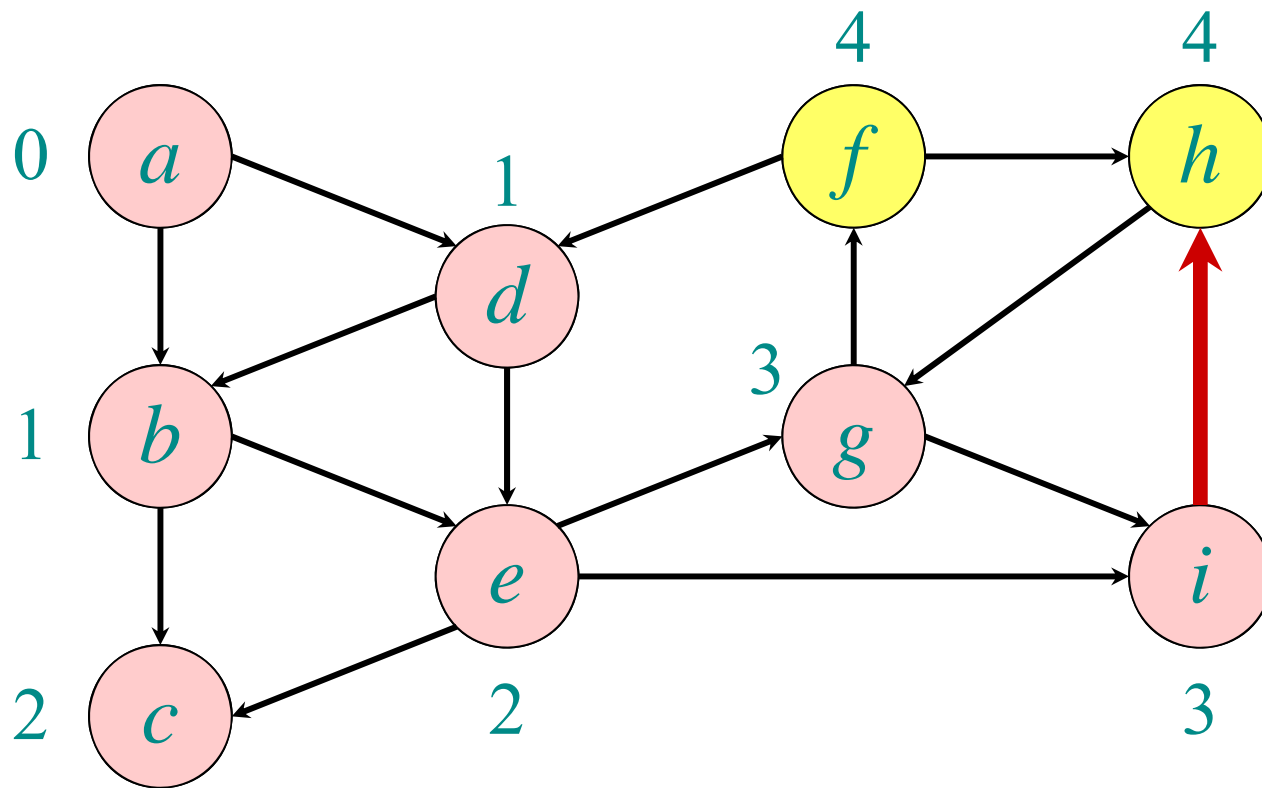


Example of breadth-first search



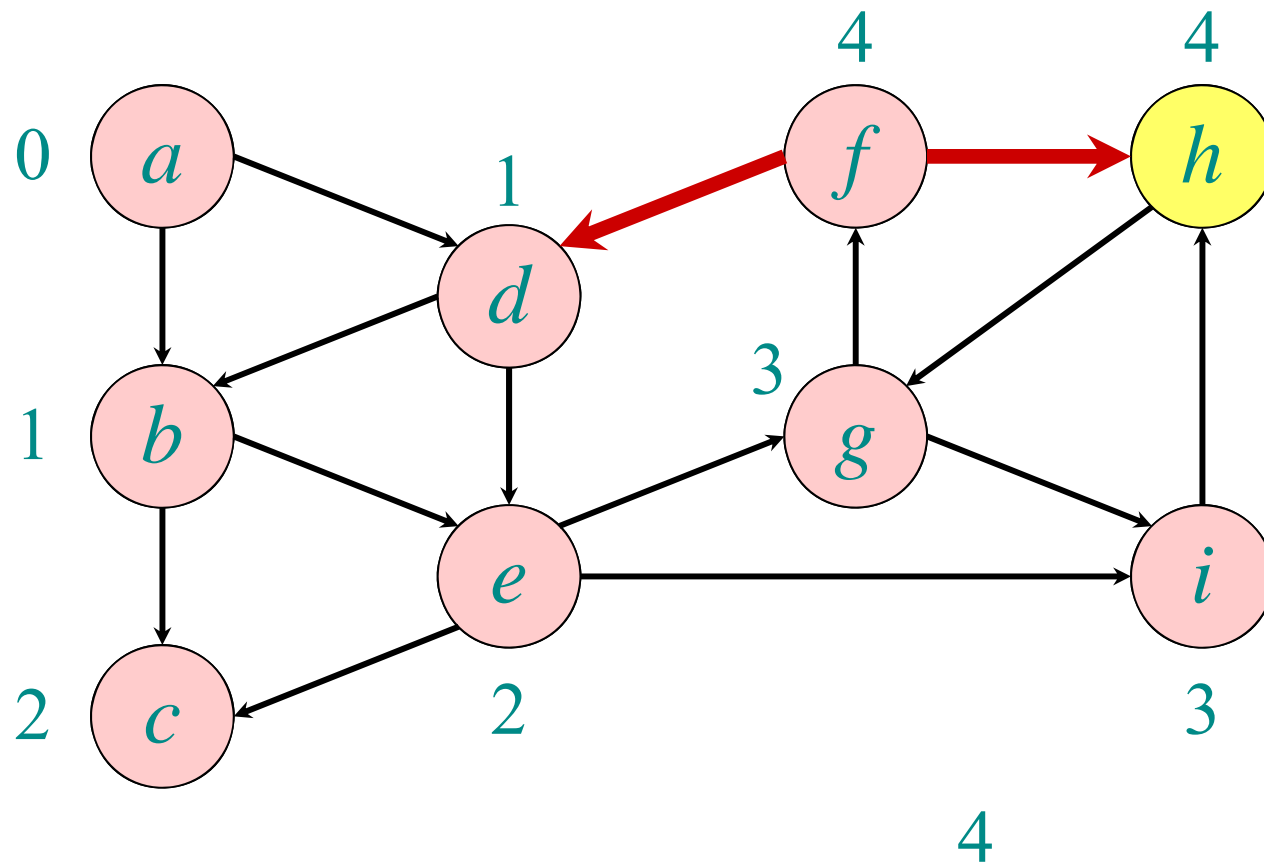
$Q: a b d c e g i f$

Example of breadth-first search



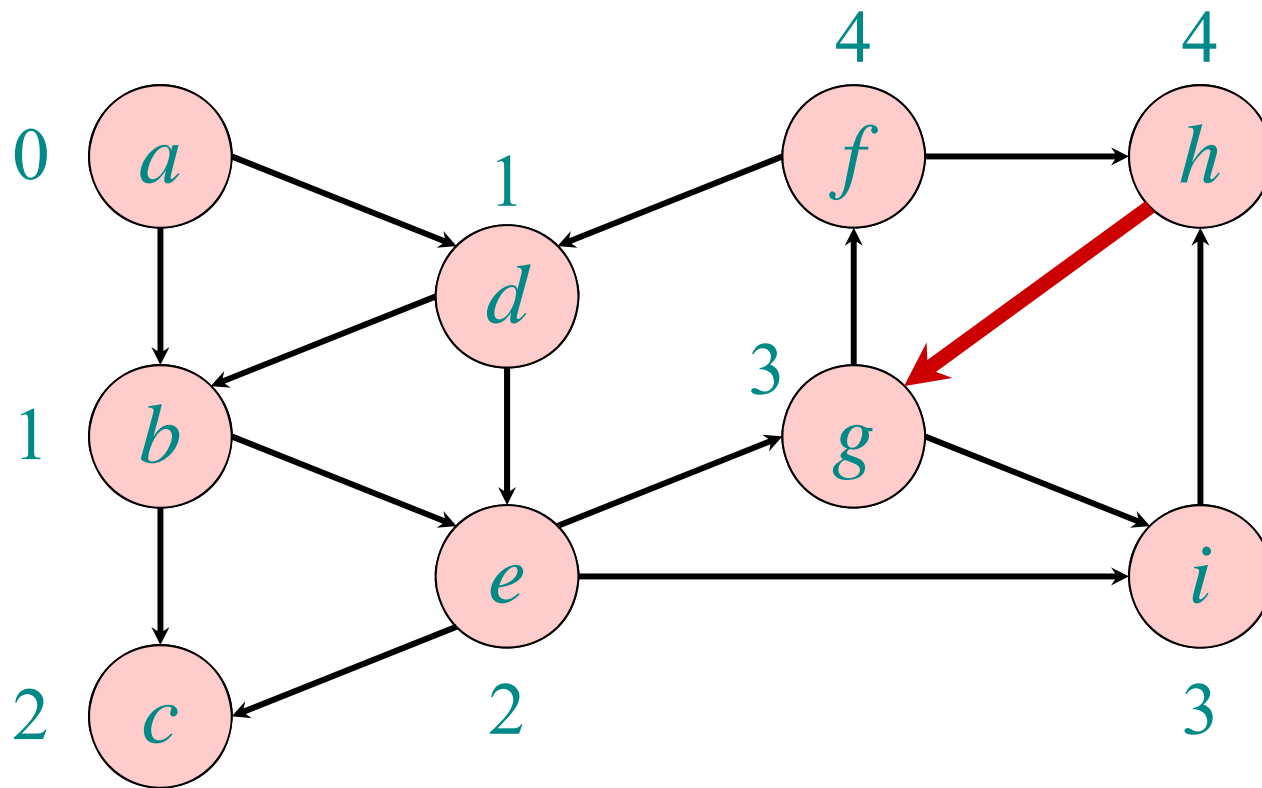
$Q: a b d c e g i f h$

Example of breadth-first search



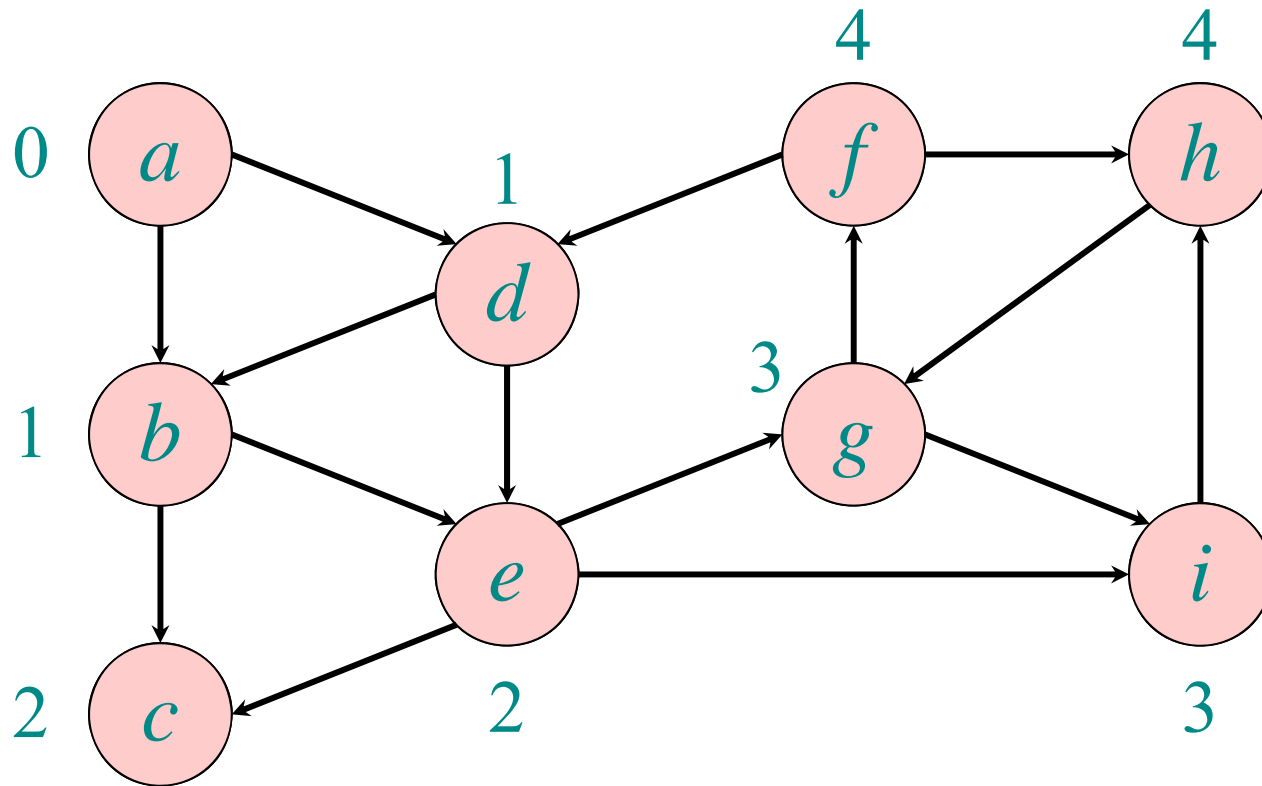
Q: *a* *b* *d* *c* *e* *g* *i* *f* *h*

Example of breadth-first search



Q: *a b d c e g i f h*

Example of breadth-first search



Q: a b d c e g i f h

Correctness of BFS

```
while  $Q \neq \emptyset$ 
do  $u \leftarrow \text{DEQUEUE}(Q)$ 
  for each  $v \in \text{Adj}[u]$ 
  do if  $d[v] = \infty$ 
    then  $d[v] \leftarrow d[u] + 1$ 
      ENQUEUE( $Q, v$ )
```

Key idea:

The FIFO Q in breadth-first search mimics the priority queue Q in Dijkstra.

- **Invariant:** v comes after u in Q implies that $d[v] = d[u]$ or $d[v] = d[u] + 1$.

How to find the actual shortest paths?

Store a predecessor tree:

$d[s] \leftarrow 0$

for each $v \in V - \{s\}$

do $d[v] \leftarrow \infty$

$S \leftarrow \emptyset$

$Q \leftarrow V$ \triangleright Q is a priority queue maintaining $V - S$

while $Q \neq \emptyset$

do $u \leftarrow \text{EXTRACT-MIN}(Q)$

$S \leftarrow S \cup \{u\}$

for each $v \in \text{Adj}[u]$

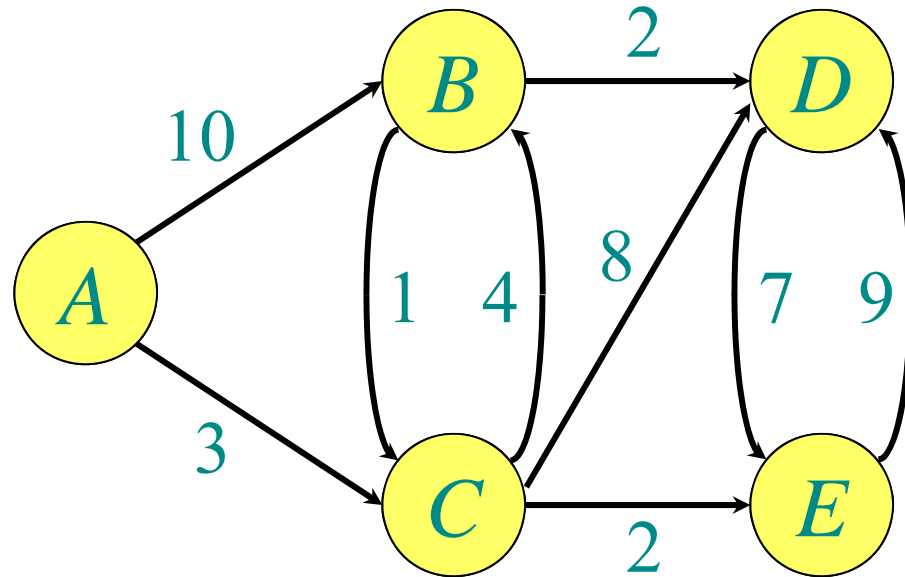
do if $d[v] > d[u] + w(u, v)$

then $d[v] \leftarrow d[u] + w(u, v)$

$\pi[v] \leftarrow u$

Example of Dijkstra's algorithm

Graph with nonnegative edge weights:



```
while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
```

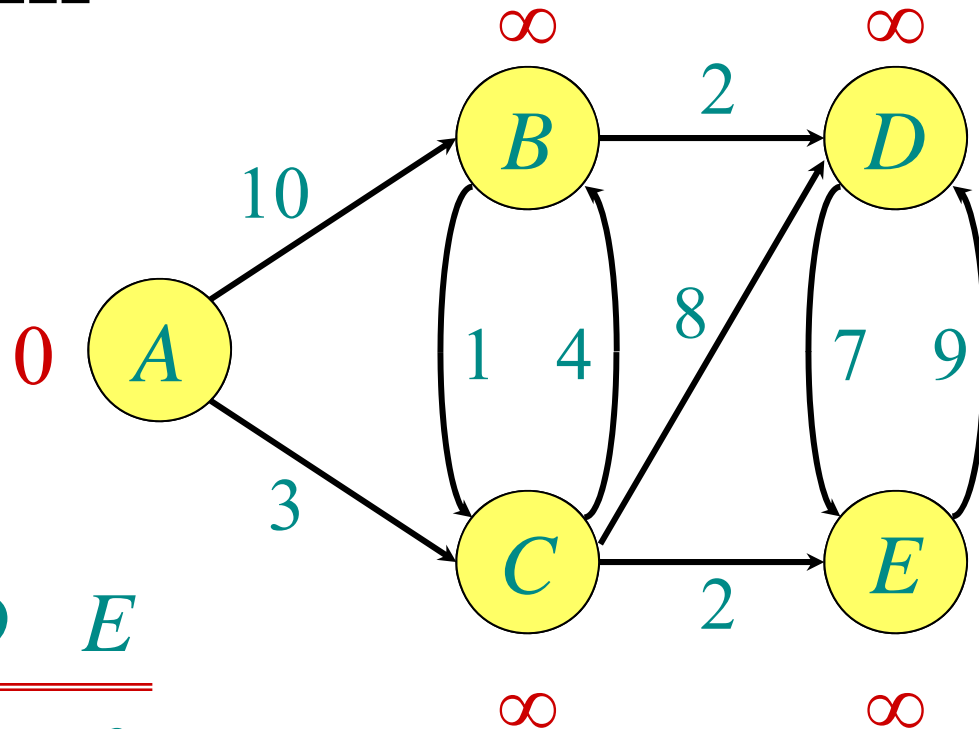
Example of Dijkstra's algorithm

Initialize:

$S: \{\}$

$Q:$

A	B	C	D	E
0	∞	∞	∞	∞



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

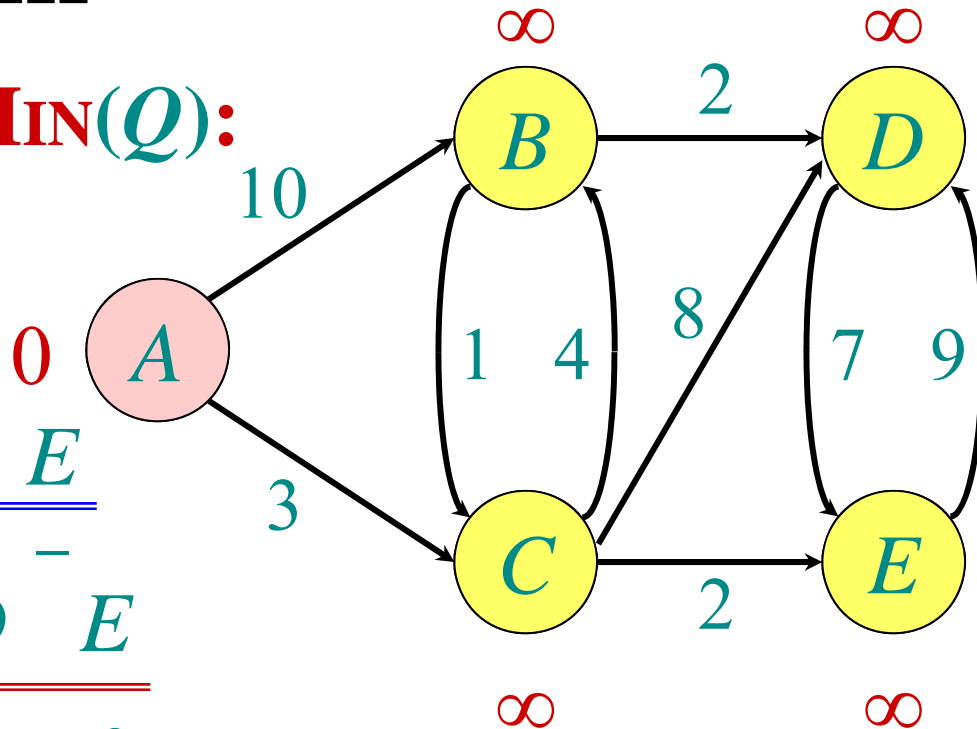
Example of Dijkstra's algorithm

“A” ← **EXTRACT-MIN**(Q):

S: { A }

π : A B C D E

Q: A B C D E
 0 ∞ ∞ ∞ ∞



```

while Q ≠ ∅ do
  u ← EXTRACT-MIN(Q)
  S ← S ∪ {u}
  for each v ∈ Adj[u] do
    if d[v] > d[u] + w(u, v) then
      d[v] ← d[u] + w(u, v)
      π[v] ← u
  
```

Example of Dijkstra's algorithm

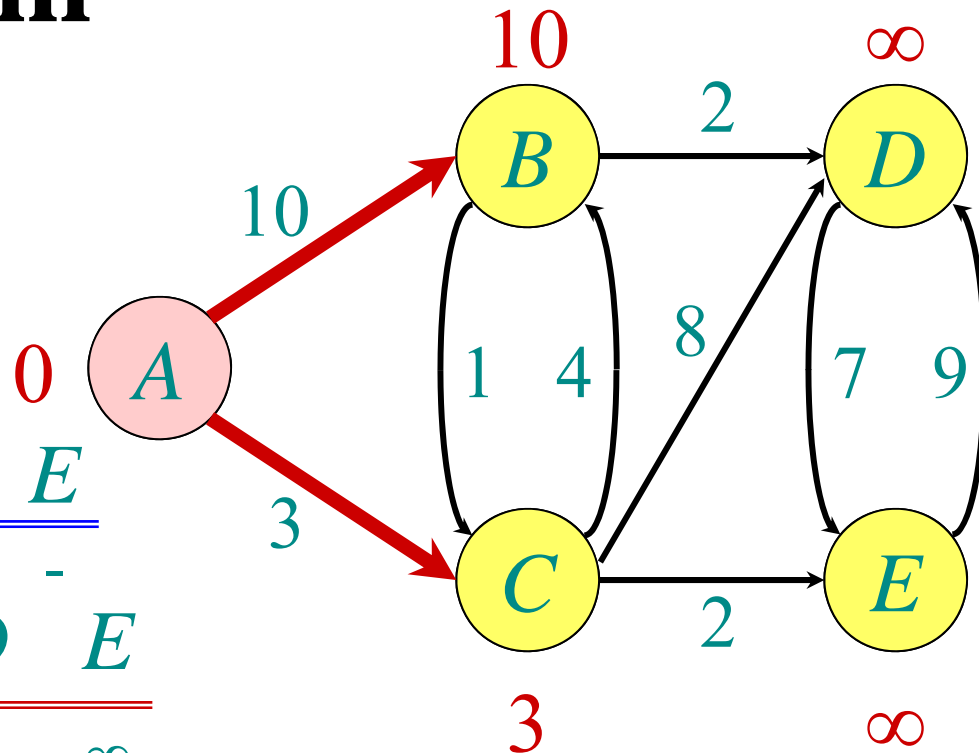
Relax all edges leaving A:

$S: \{A\}$

$\pi: \underline{A \quad B \quad C \quad D \quad E}$

$Q: \underline{A \quad B \quad C \quad D \quad E}$

0	∞	∞	∞	∞
	10	3	-	-



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

Example of Dijkstra's algorithm

Relax all edges leaving A:

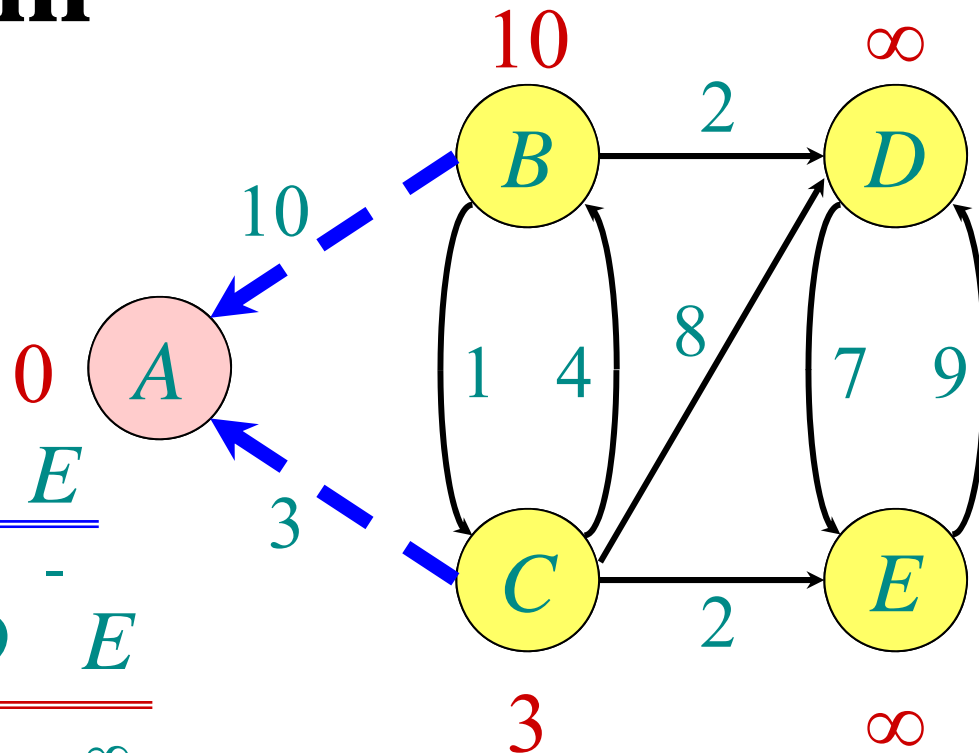
$S: \{A\}$

$\pi:$ A B C D E

- A A - -

$Q:$ A B C D E

0	∞	∞	∞	∞
	10	3	-	-



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

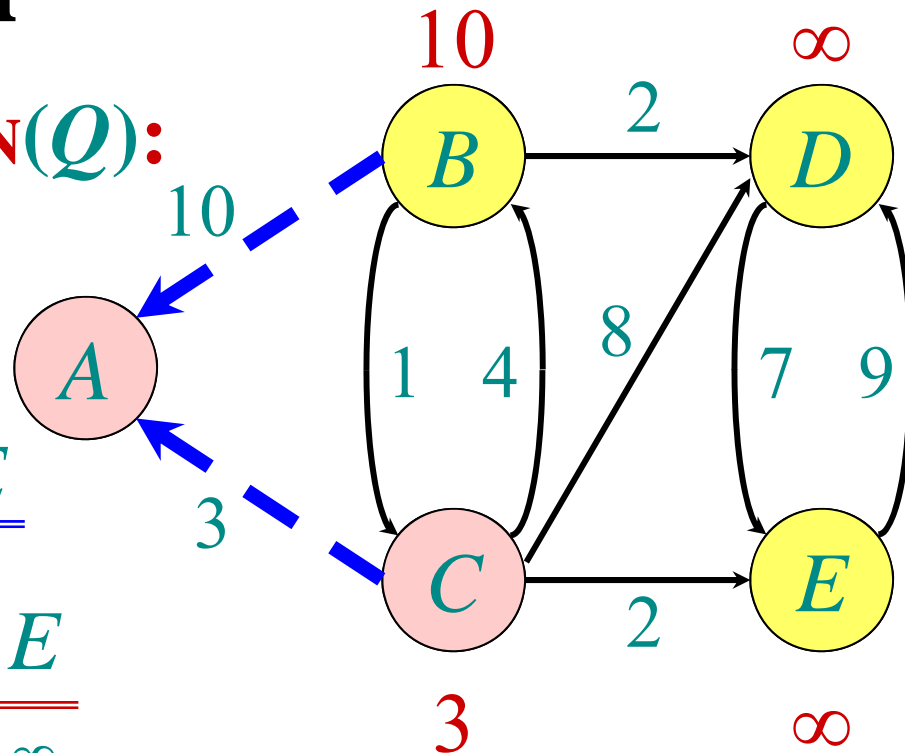
Example of Dijkstra's algorithm

“C” ← **EXTRACT-MIN(Q)**:

$S: \{A, C\}$

$\pi:$ A B C D E
 - A A - -

$Q:$	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	-	-



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

Example of Dijkstra's algorithm

Relax all edges leaving C :

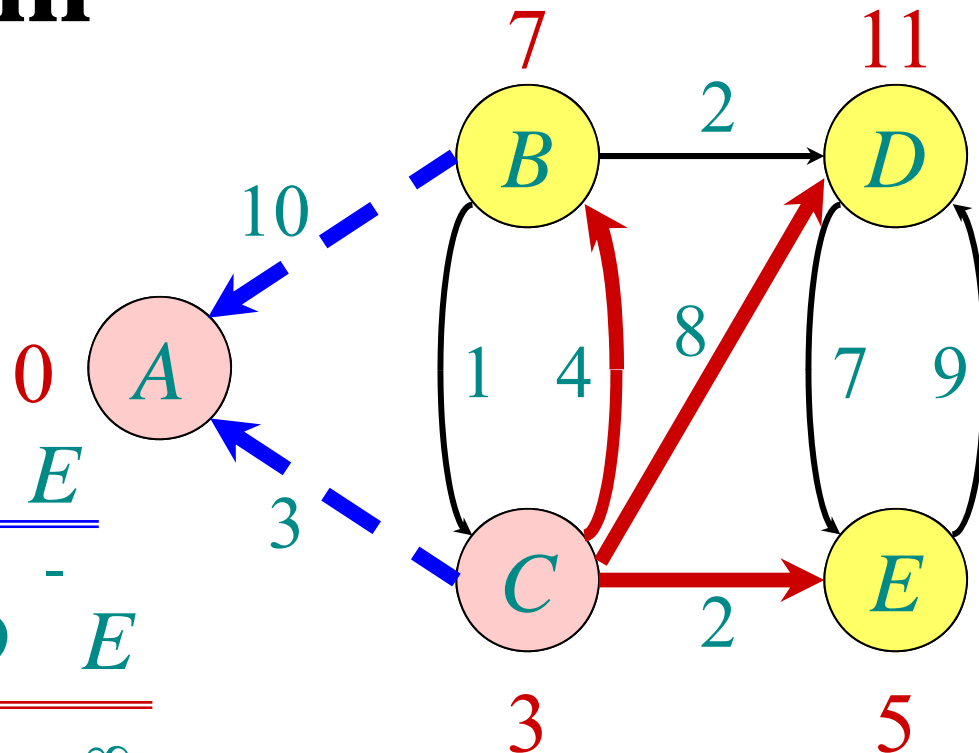
$S: \{A, C\}$

$\pi:$

A	B	C	D	E
-	A	A	-	-

$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	-	-
	7		11	5



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```


Example of Dijkstra's algorithm

Relax all edges leaving C :

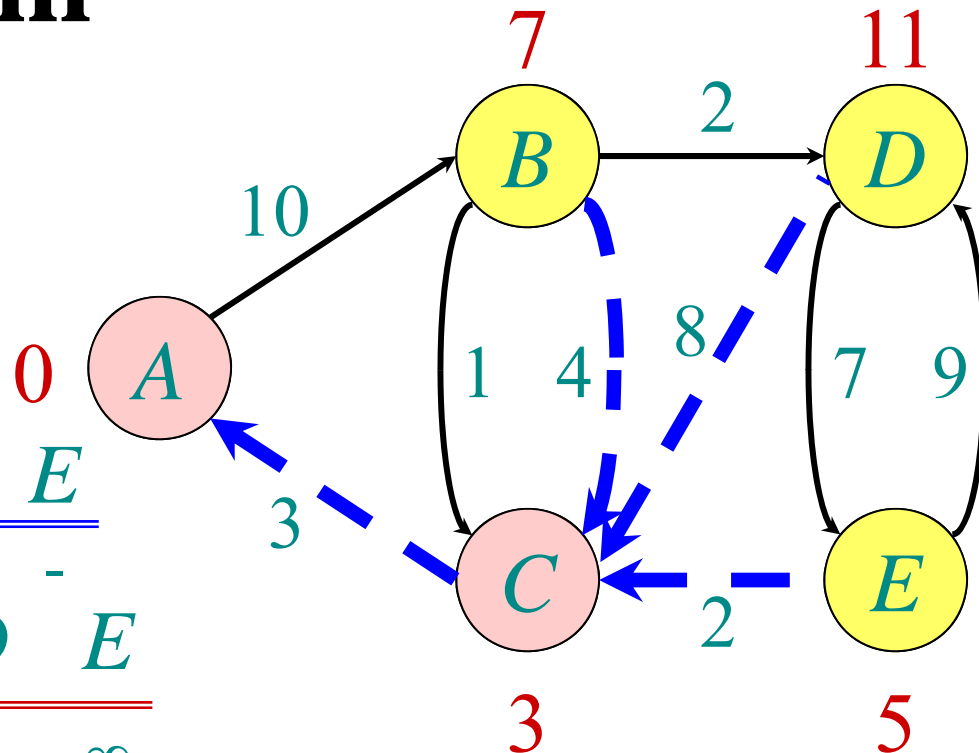
$S: \{A, C\}$

$\pi:$

A	B	C	D	E
-	A	A	-	-

$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	-	-
	7		11	5



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

Example of Dijkstra's algorithm

“E” ← **EXTRACT-MIN(Q)**:

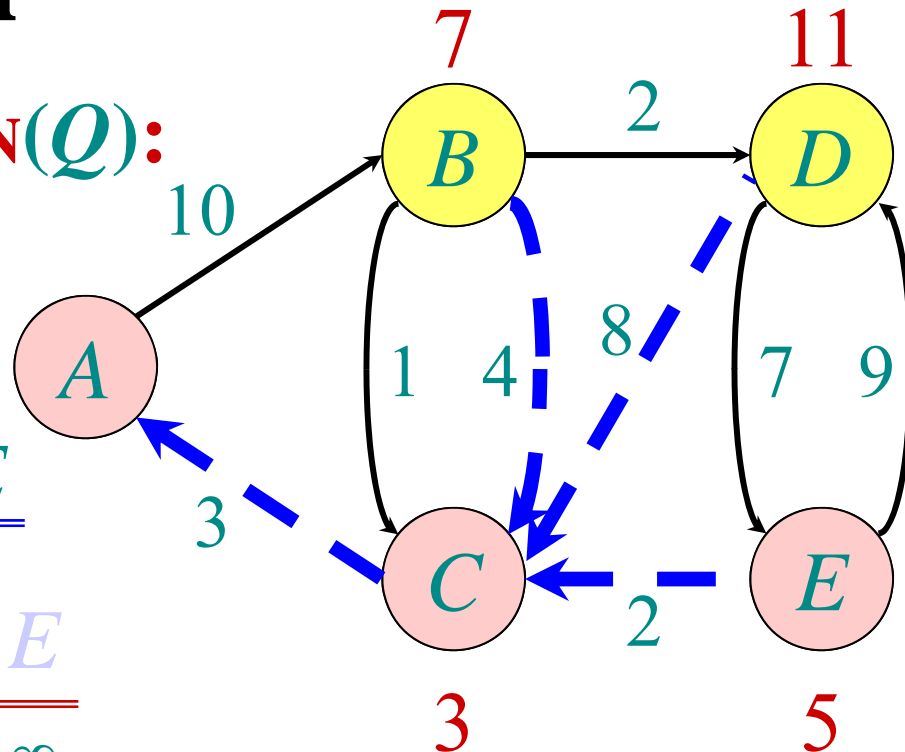
$S: \{A, C, E\}$

$\pi:$

A	B	C	D	E
-	C	A	C	C

$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	-	-
	7		11	5



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

Example of Dijkstra's algorithm

Relax all edges leaving E :

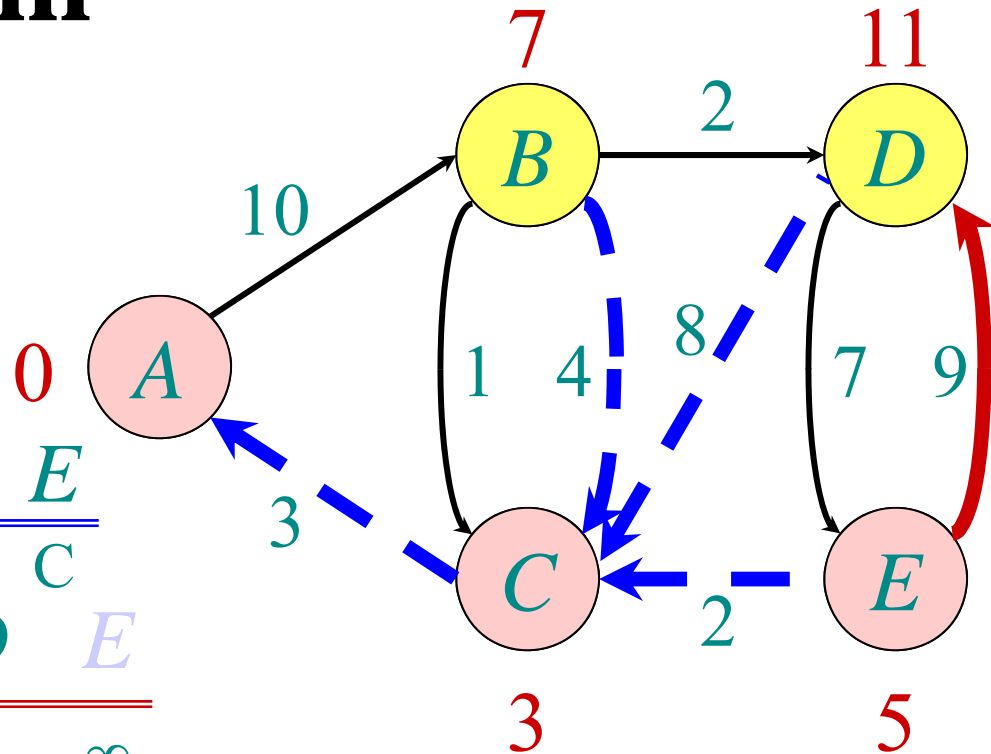
$S: \{A, C, E\}$

$\pi:$

<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
-	C	A	C	C

$Q:$

A	B	C	D	E
0	∞	∞	∞	∞
	10	3	∞	∞
	7		11	5
	7		11	



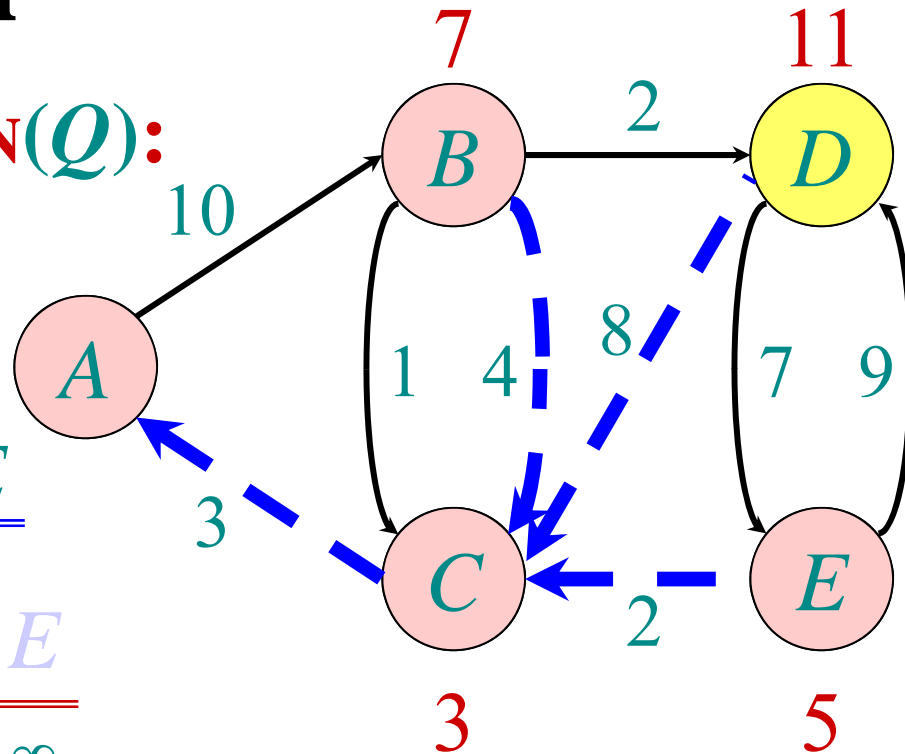
```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

Example of Dijkstra's algorithm

“B” ← EXTRACT-MIN(Q):

$S: \{A, C, E, B\}$ 0
 $\pi:$ A B C D E
 - C A C C
 $Q:$ A B C D E
 0 ∞ ∞ ∞ ∞
 10 3 ∞ ∞ ∞
 7 7 11 5
 7 11



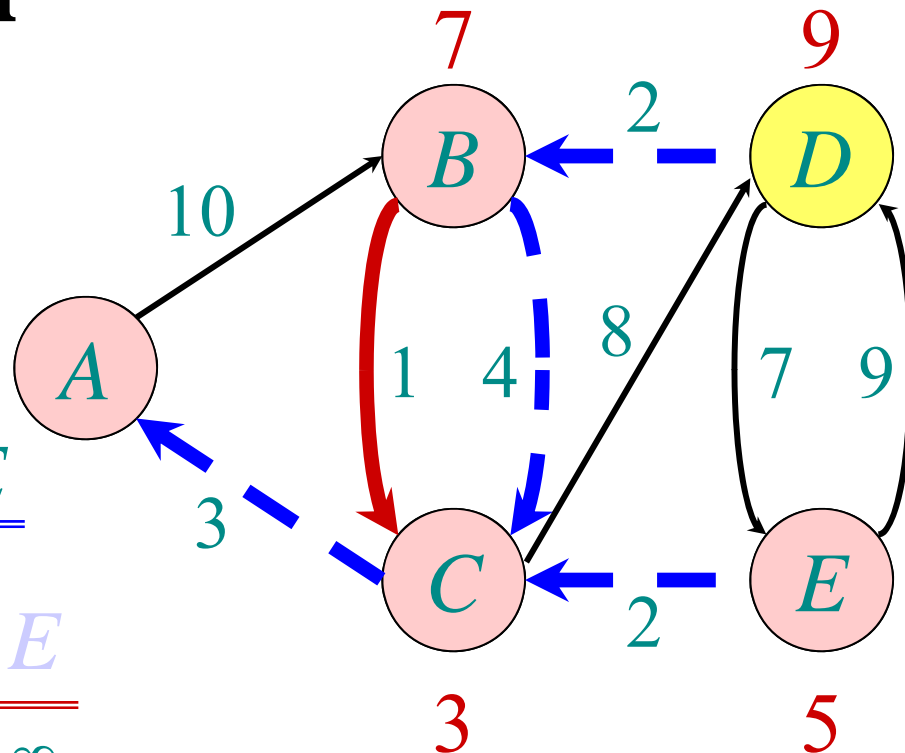
```

while Q ≠ ∅ do
    u ← EXTRACT-MIN(Q)
    S ← S ∪ {u}
    for each v ∈ Adj[u] do
        if d[v] > d[u] + w(u, v) then
            d[v] ← d[u] + w(u, v)
            π[v] ← u
    
```

Example of Dijkstra's algorithm

Relax all edges leaving B :

$S: \{ A, C, E, B \}$ 0
 $\pi:$ A B C D E
 - C A B C
 $Q:$ A B C D E
 0 ∞ ∞ ∞ ∞
 10 3 ∞ ∞
 7 11 5
 7 11
 9



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```

Example of Dijkstra's algorithm

“D” ← **EXTRACT-MIN(Q)**:

$S: \{A, C, E, B, D\}$

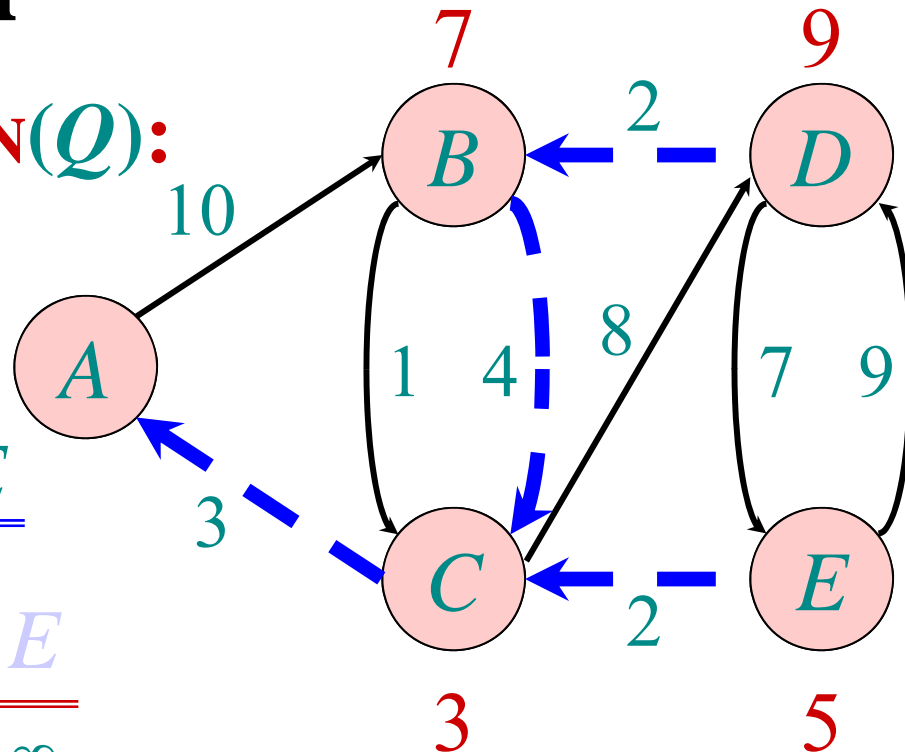
$\pi:$

A	B	C	D	E
-	C	A	B	C

$Q:$

A	B	C	D	E
---	---	---	---	---

0	∞	∞	∞	∞
10	3	∞	∞	∞
7		11	5	
7		11		
		9		



```

while  $Q \neq \emptyset$  do
   $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
   $S \leftarrow S \cup \{u\}$ 
  for each  $v \in \text{Adj}[u]$  do
    if  $d[v] > d[u] + w(u, v)$  then
       $d[v] \leftarrow d[u] + w(u, v)$ 
       $\pi[v] \leftarrow u$ 
  
```