CMPS 2200 – Fall 2014

Randomized Algorithms, Quicksort and Randomized Selection

Carola Wenk

Slides courtesy of Charles Leiserson with additions by Carola Wenk

Deterministic Algorithms

Runtime for deterministic algorithms with input size n:

- Best-case runtime
  - ➔ Attained by one input of size n
- Worst-case runtime
  - ➔ Attained by one input of size n
- Average runtime
  - ➔ Averaged **over all possible inputs** of size n

Deterministic Algorithms: Insertion Sort

```
for j=2 to n {
   key = A[j]
   // insert A[j] into sorted sequence A[1..j-1]
   i=j-1
   while(i>0 && A[i]>key){
     A[i+1]=A[i]
     i--
   }
   A[i+1]=key
}
```

- Best case runtime?
- Worst case runtime?

Deterministic Algorithms: Insertion Sort

Best-case runtime: $O(n)$, input $[1,2,3,\ldots,n]$

➔Attained by one input of size n

- Worst-case runtime: $O(n^2)$, input $[n, n\text{-}1, \ldots,2,1]$

➔ Attained by one input of size n

- Average runtime : $O(n^2)$

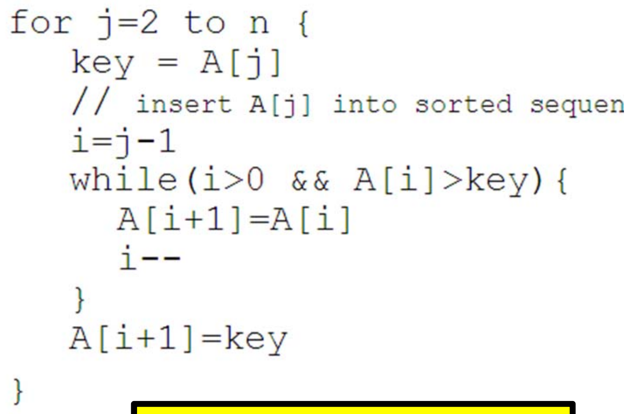
➔ Averaged **over all possible inputs** of size n

- What kind of inputs are there?
- How many inputs are there?

Average Runtime

- What kind of inputs are there?
  - Do $[1,2,\ldots,n]$ and $[5,6,\ldots,n+5]$ cause different behavior of Insertion Sort?
  - No. Therefore it suffices to only consider all permutations of $[1,2,\ldots,n]$.
- How many inputs are there?
  - There are $n!$ different permutations of $[1,2,\ldots,n]$

Average Runtime Insertion Sort: $n=4$



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    // insert A[j] into sorted sequen
    i=j-1
    while(i>0 && A[i]>key){
       A[i+1]=A[i]
       i--
    }
    A[i+1]=key
}
```

- Inputs: 4!=24

[1,2,3,4] **0**	[4,1,2,3] **3**	[4,1,3,2] **4**	[4,3,2,1] **6**
[2,1,3,4] **1**	[1,4,2,3] **2**	[1,4,3,2] **3**	[3,4,2,1] **5**
[1,3,2,4] **1**	[1,2,4,3] **1**	[1,3,4,2] **2**	[3,2,4,1] **4**
[3,1,2,4] **2**	[4,2,1,3] **4**	[4,3,1,2] **5**	[4,2,3,1] **5**
[3,2,1,4] **3**	[2,1,4,3] **2**	[3,4,1,2] **4**	[2,4,3,1] **4**
[2,3,1,4] **2**	[2,4,1,3] **3**	[3,1,4,2] **3**	[2,3,4,1] **3**

- Runtime is proportional to: 3 + **#times in while loop**
- Best: 3+**0**, Worst: 3+**6**=9, Average: 3+**72**/24 = 6

Average Runtime: Insertion Sort

- The average runtime averages runtimes over all $n!$ different input permutations
- Disadvantage of considering average runtime:
  - There are still worst-case inputs that will have the worst-case runtime
  - Are all inputs really equally likely? That depends on the application

$\Rightarrow$ **Better:** Use a randomized algorithm

Randomized Algorithm: Insertion Sort

- **Randomize the order of the input array:**
  - Either prior to calling insertion sort,
  - or during insertion sort (insert random element)
- This makes the runtime depend on a probabilistic experiment (sequence of numbers obtained from random number generator; or random input permutation)

  ⇒Runtime is a random variable (maps sequence of random numbers to runtimes)
- **Expected runtime** = expected value of runtime random variable

Randomized Algorithm: Insertion Sort

- Runtime is independent of input order ([1,2,3,4] may have good or bad runtime, depending on sequence of random numbers)
- No assumptions need to be made about input distribution
- No one specific input elicits worst-case behavior
- The worst case is determined only by the output of a random-number generator.

⇒ When possible use expected runtimes of randomized algorithms instead of average case analysis of deterministic algorithms

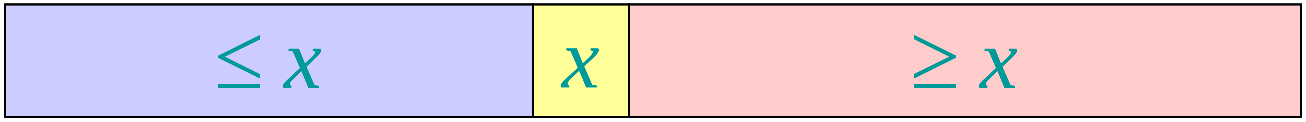
Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).
- We are going to perform an expected runtime analysis on randomized quicksort

Quicksort: Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



$\leq x$	x	$\geq x$

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

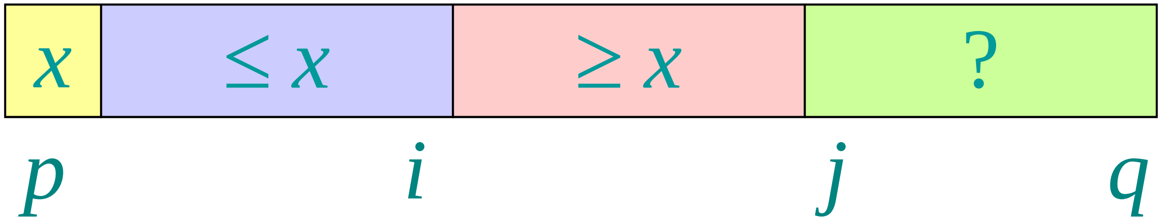
Key: *Linear-time partitioning subroutine.*

Partitioning subroutine

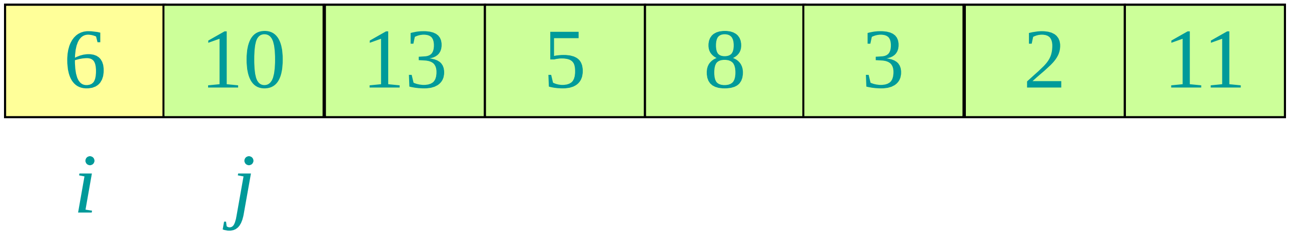
```
PARTITION($A, p, q$)  ⊳ $A[p\,.\,.\,q]$
    $x \leftarrow A[p]$          ⊳ pivot = $A[p]$
    $i \leftarrow p$
    for $j \leftarrow p + 1$ to $q$
        do if $A[j] \leq x$
            then  $i \leftarrow i + 1$
                  exchange $A[i] \leftrightarrow A[j]$
    exchange $A[p] \leftrightarrow A[i]$
    return $i$
```

Running time $= O(n)$ for n elements.

Invariant:

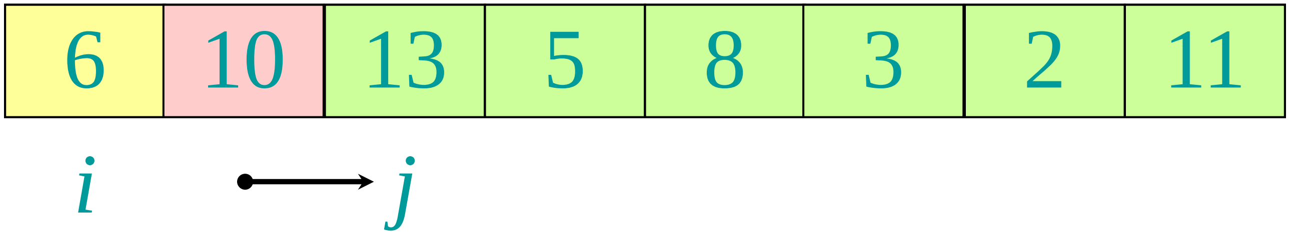


Example of partitioning



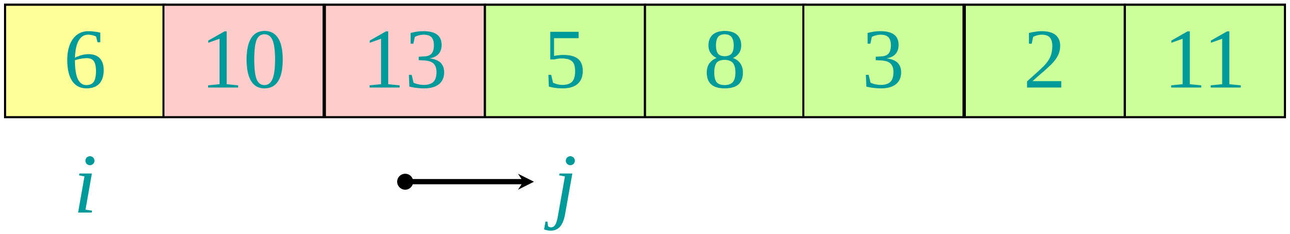
6	10	13	5	8	3	2	11
i	j						

Example of partitioning



6	10	13	5	8	3	2	11
i		*j*					

Example of partitioning



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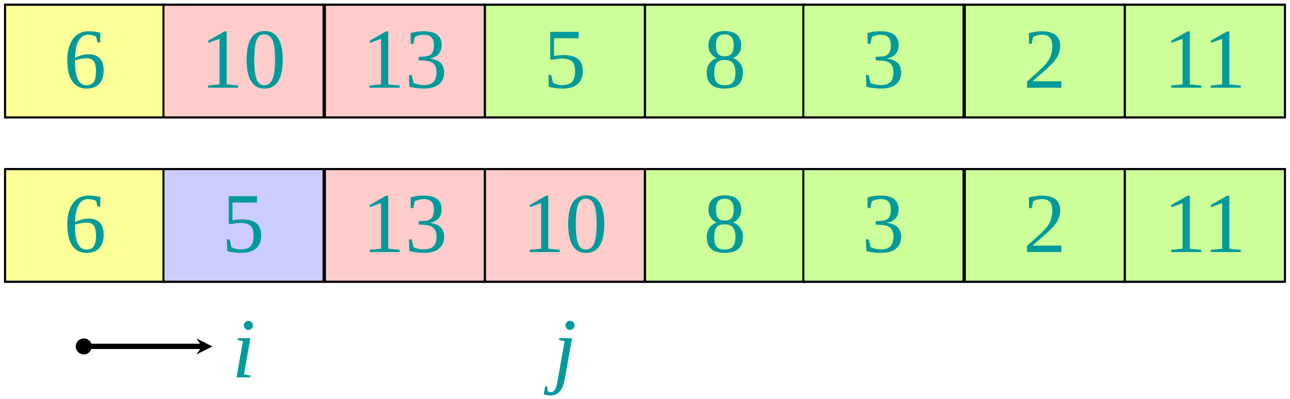
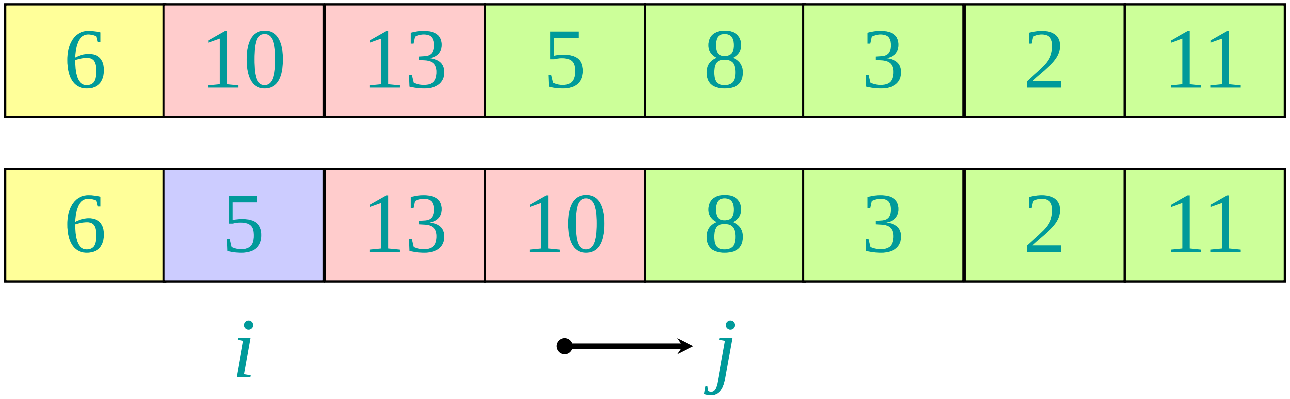


6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
→	i		j				

Example of partitioning



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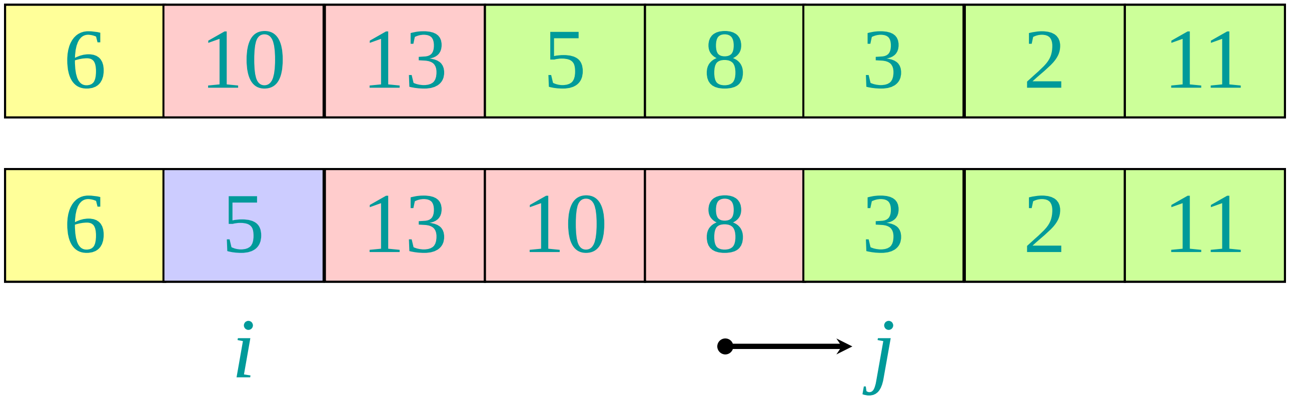
Example of partitioning



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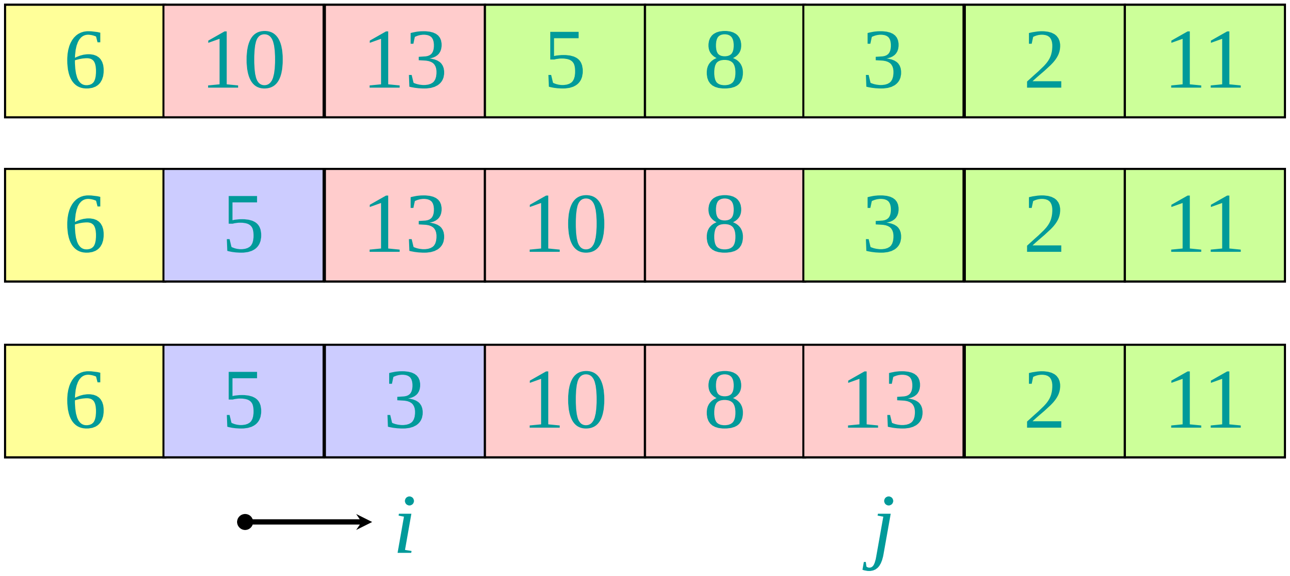
Example of partitioning



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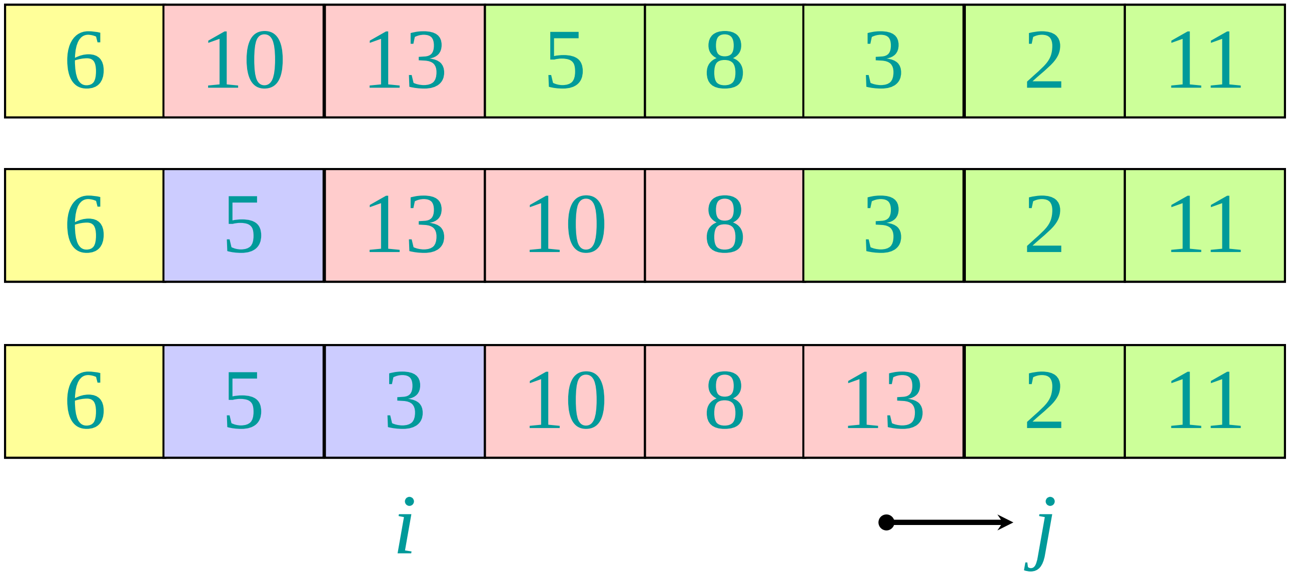
6	5	3	10	8	13	2	11

Example of partitioning

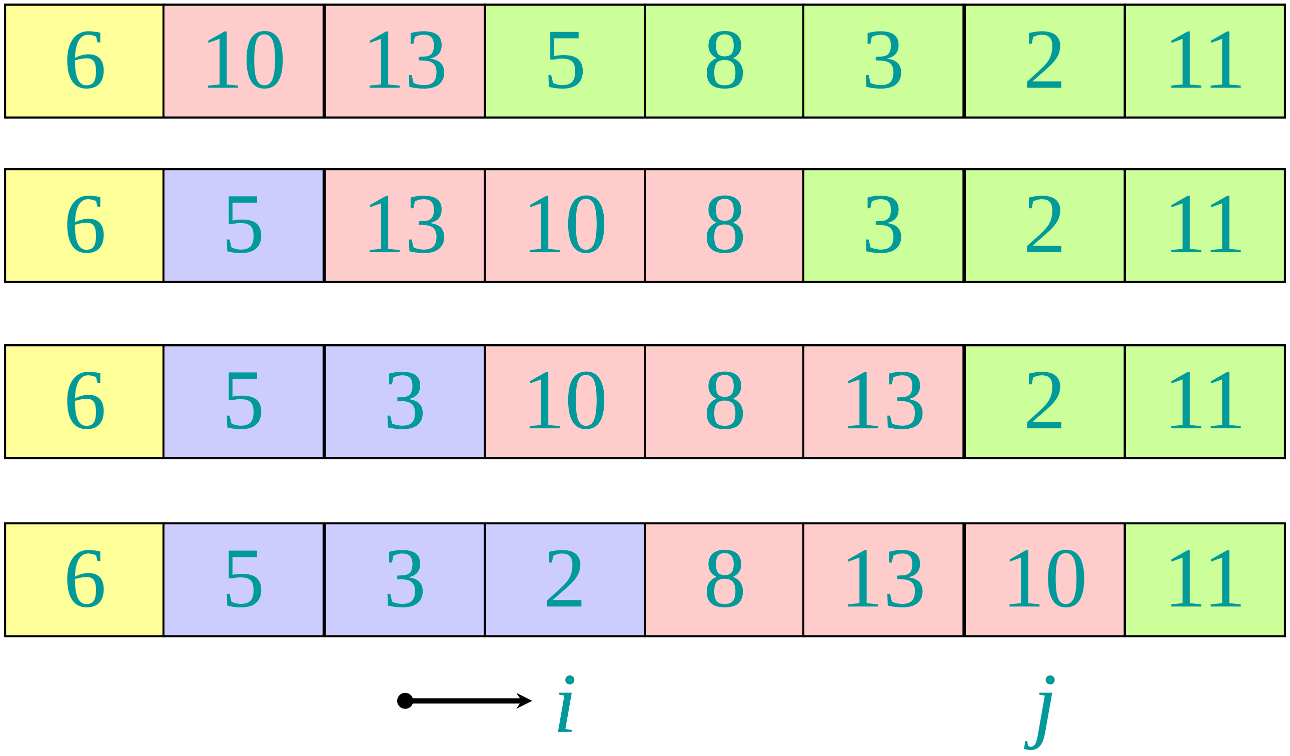


6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

Example of partitioning



6	10	13	5	8	3	2	11
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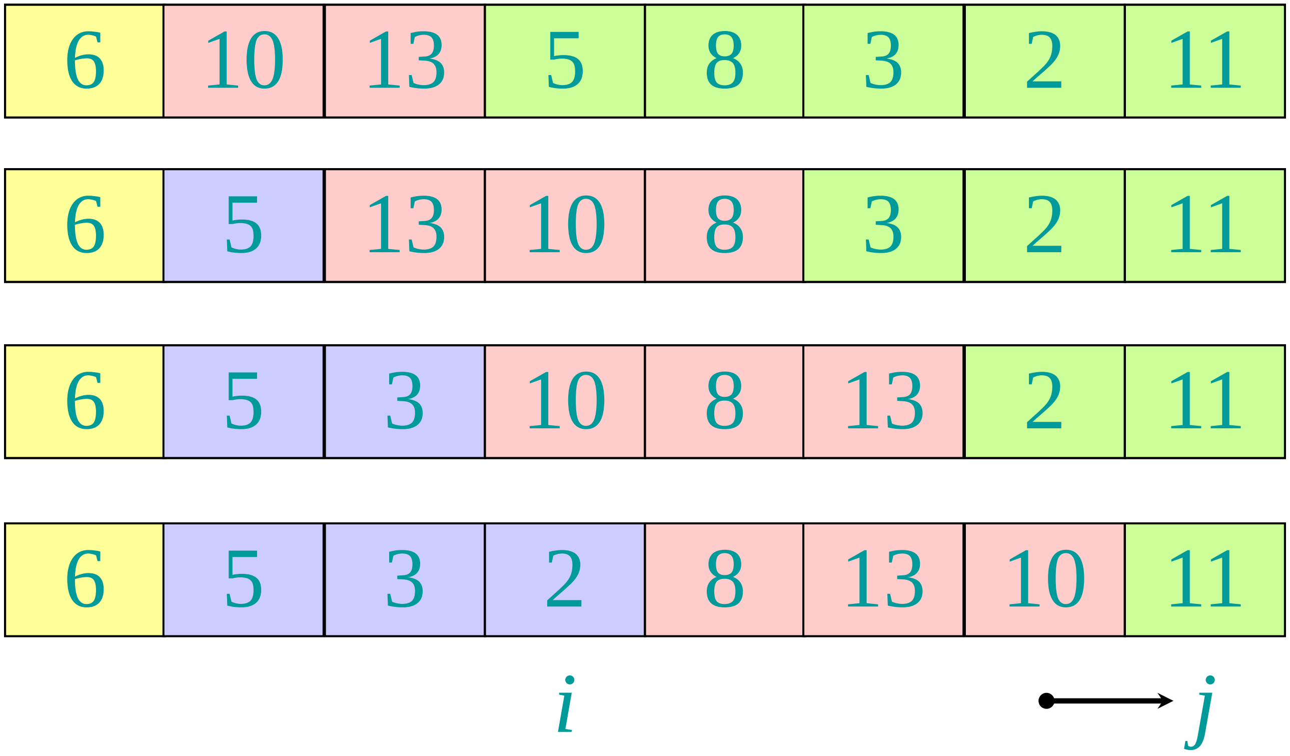
Example of partitioning



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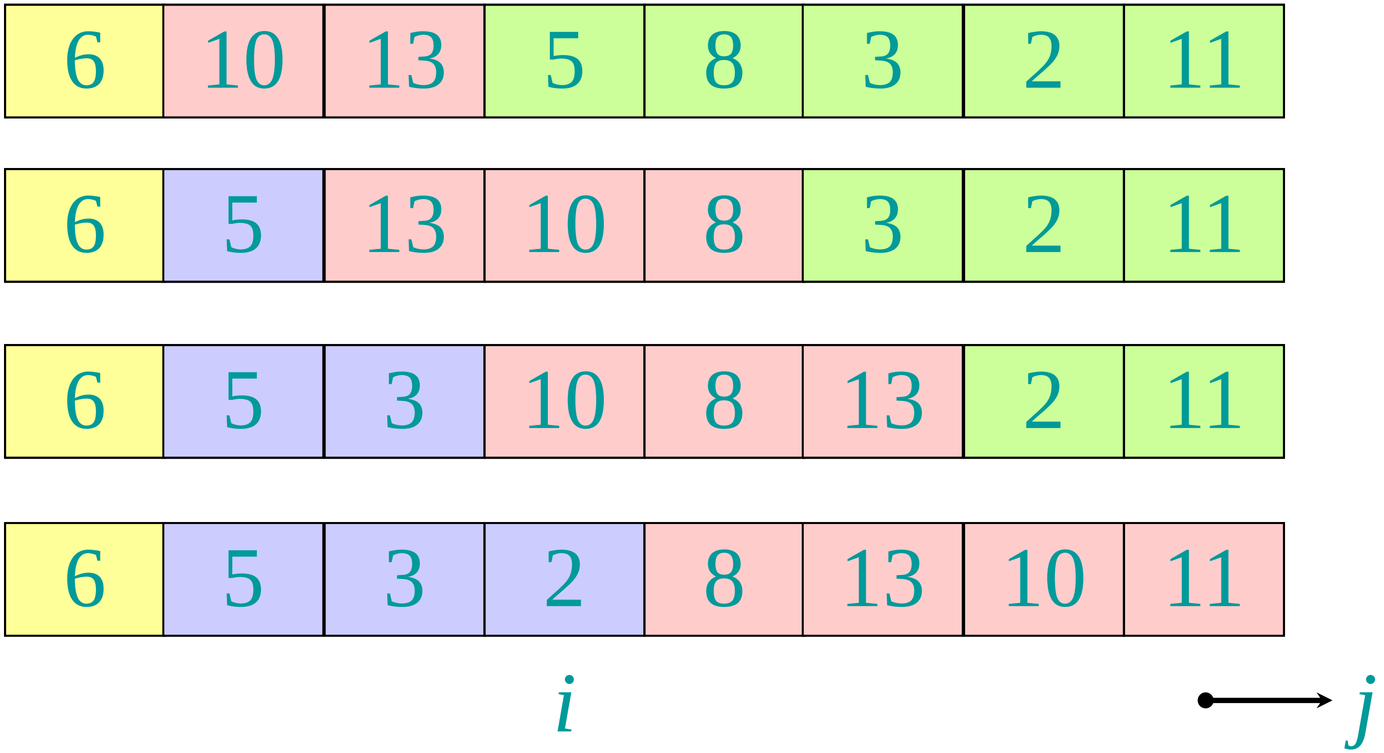

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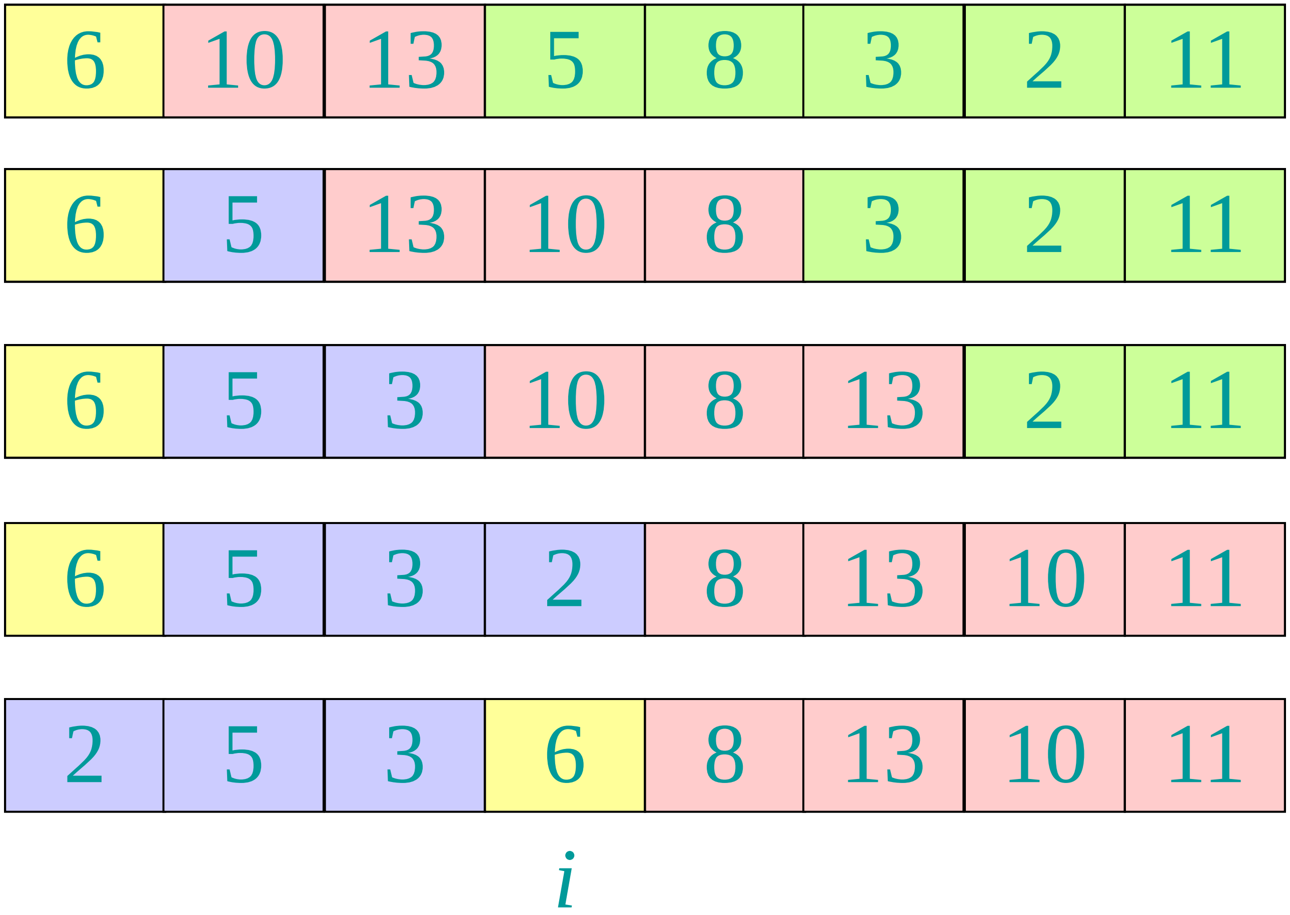

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11

Pseudocode for quicksort

```
QUICKSORT($A, p, r$)
    if $p < r$
        then $q \leftarrow$ PARTITION($A, p, r$)
            QUICKSORT($A, p, q-1$)
            QUICKSORT($A, q+1, r$)
```

Initial call: QUICKSORT(A, 1, n)

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

```
QUICKSORT($A, p, r$)
    if $p < r$
        then $q \leftarrow$ PARTITION($A, p, r$)
            QUICKSORT($A, p, q-1$)
            QUICKSORT($A, q+1, r$)
```

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \qquad \textit{(arithmetic series)} \end{aligned}$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

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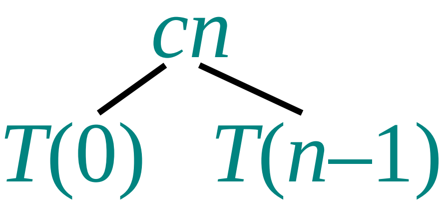
$T(n)$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

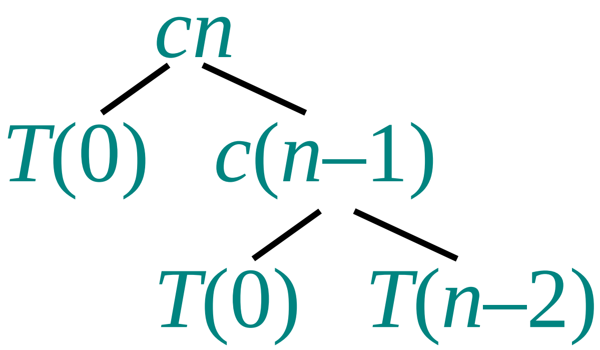


cn

$T(0)$ $T(n-1)$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



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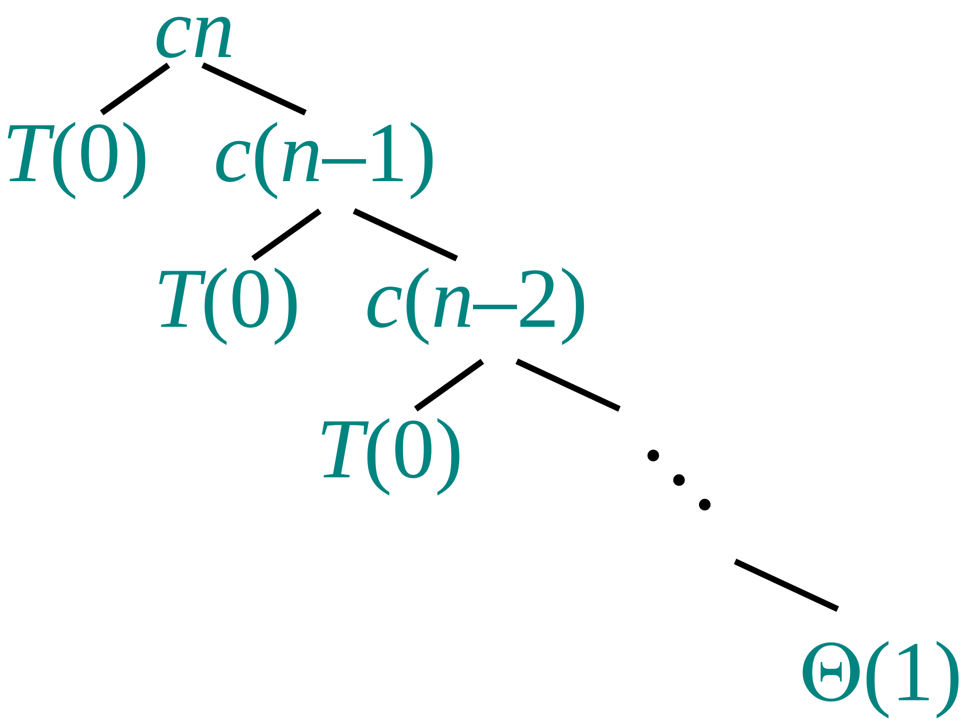
$T(0)$ $c(n-1)$

$T(0)$ $T(n-2)$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



cn

$T(0)$ $c(n-1)$

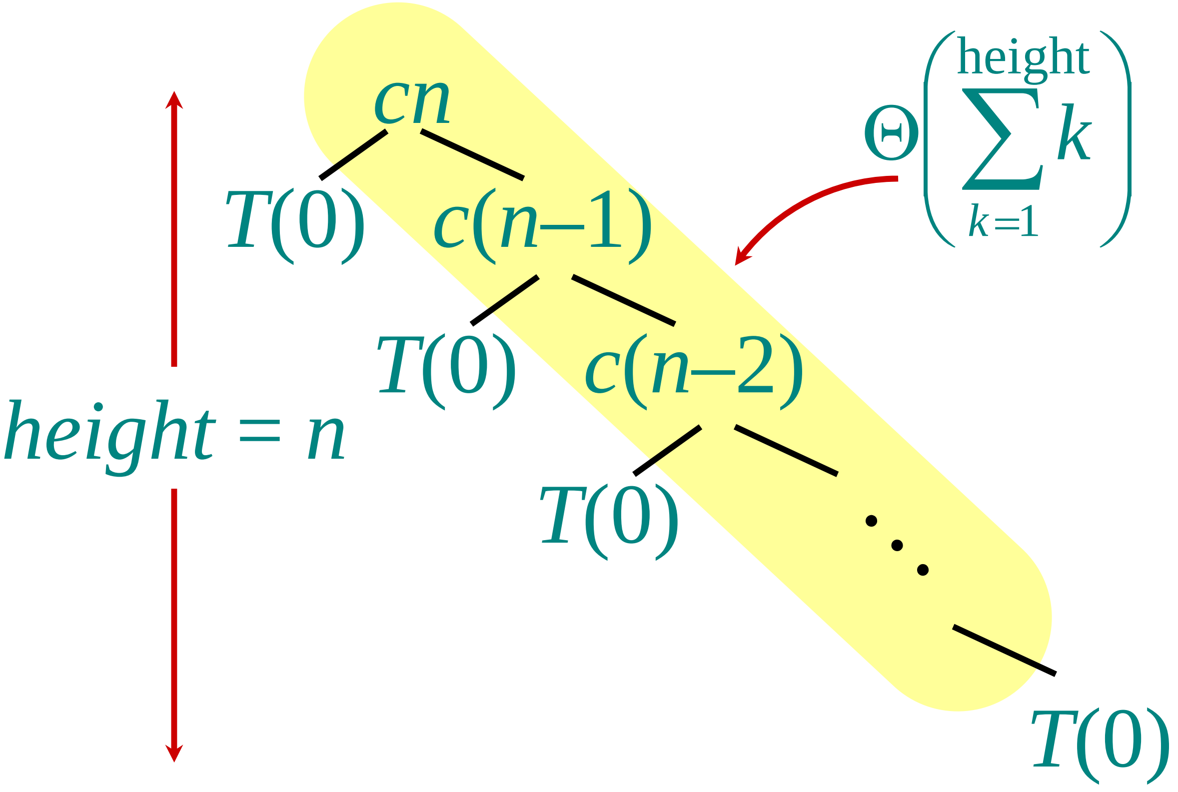
$T(0)$ $c(n-2)$

$T(0)$

$\Theta(1)$

Worst-case recursion tree

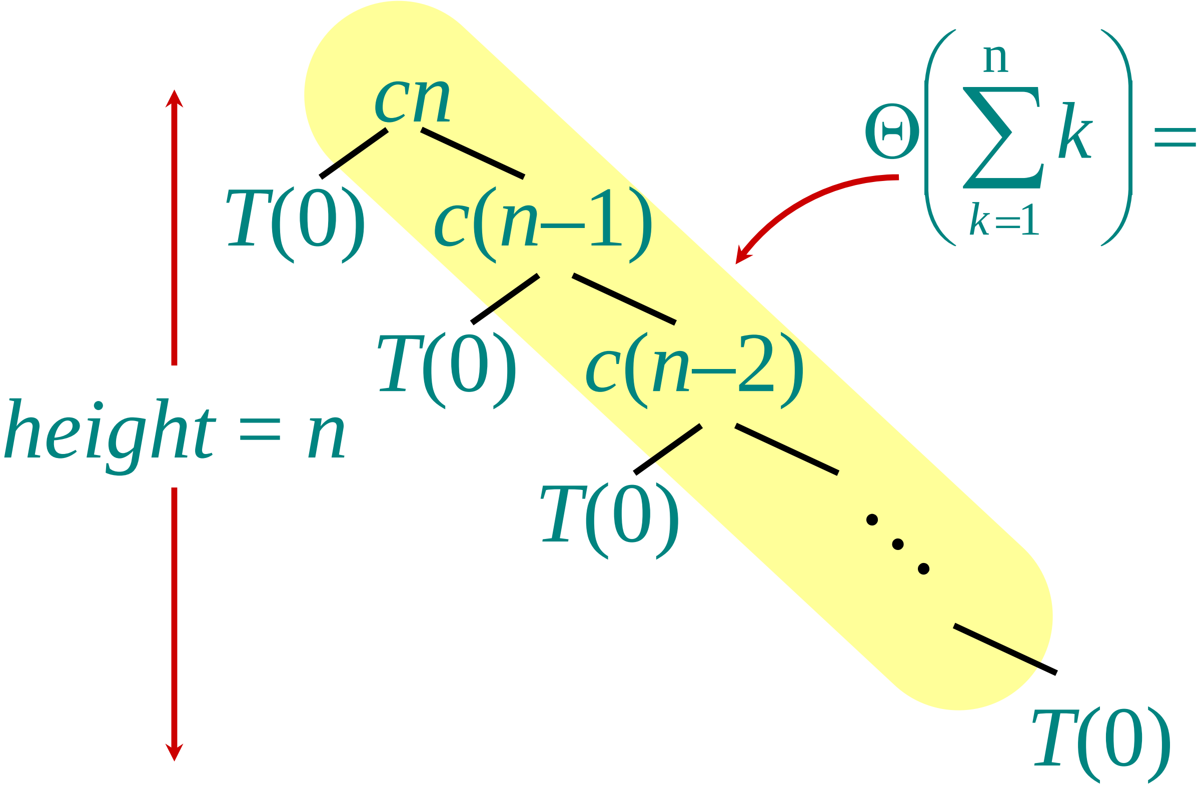
$$T(n) = T(0) + T(n–1) + cn$$



$$\Theta\left(\sum_{k=1}^{\text{height}} k\right)$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



cn

$T(0)$ $c(n-1)$

$T(0)$ $c(n-2)$

$T(0)$ …

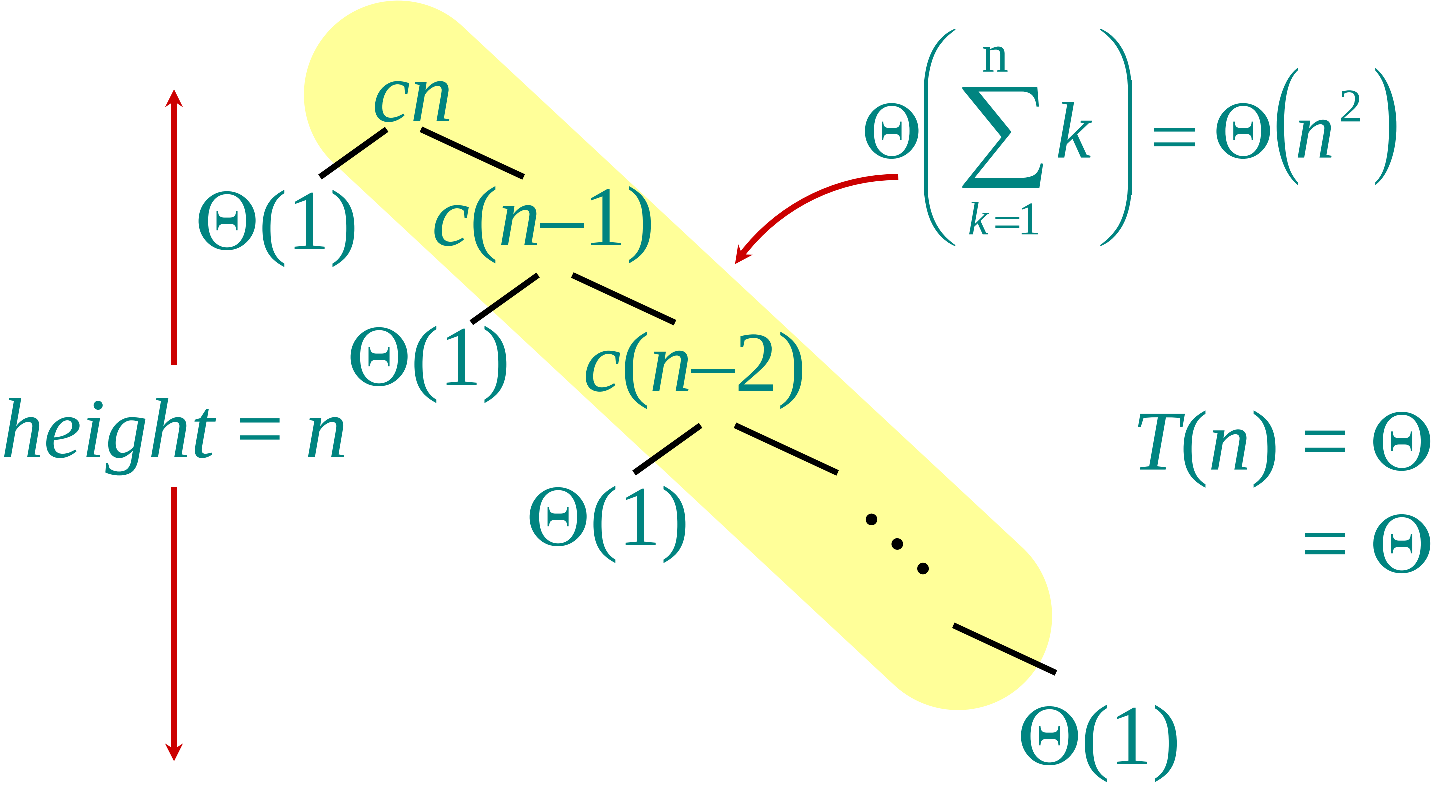
$T(0)$

$height = n$

$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



cn

$\Theta(1)$ $c(n-1)$

$\Theta(1)$ $c(n-2)$

$\Theta(1)$ $\ldots$

$\Theta(1)$

$height = n$

$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

$$\begin{aligned} T(n) &= \Theta(n) + \Theta(n^2) \\ &= \Theta(n^2) \end{aligned}$$

Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \log n) \end{aligned}$$

(same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

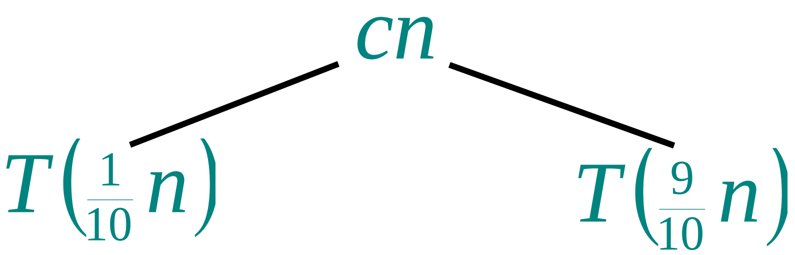
$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

Analysis of “almost-best” case

$T(n)$

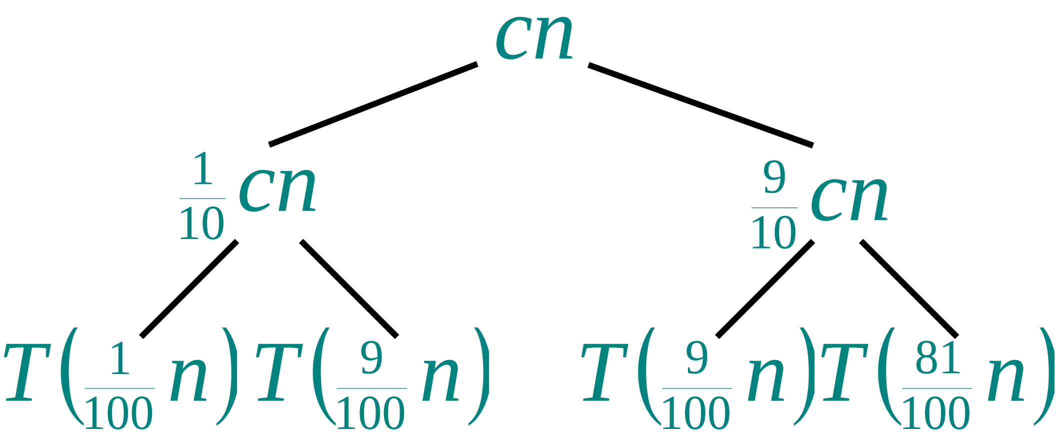
Analysis of “almost-best” case



cn

$T\left(\frac{1}{10}n\right)$ $T\left(\frac{9}{10}n\right)$

Analysis of “almost-best” case

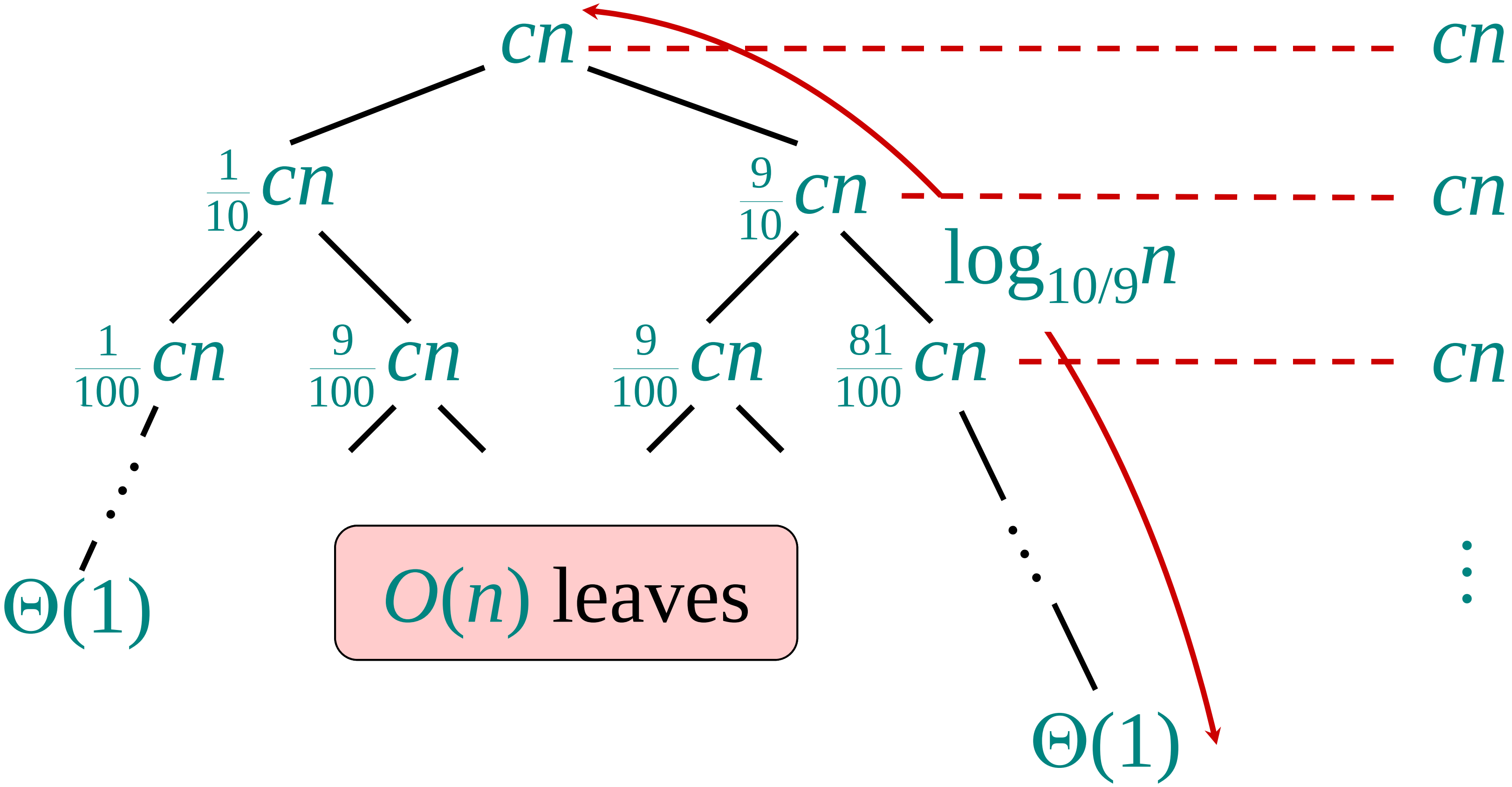


$$cn$$

$$\tfrac{1}{10}cn \qquad \tfrac{9}{10}cn$$

$$T\left(\tfrac{1}{100}n\right)\; T\left(\tfrac{9}{100}n\right) \qquad T\left(\tfrac{9}{100}n\right) T\left(\tfrac{81}{100}n\right)$$

Analysis of “almost-best” case



cn

$\frac{1}{10}cn$ $\frac{9}{10}cn$

$\frac{1}{100}cn$ $\frac{9}{100}cn$ $\frac{9}{100}cn$ $\frac{81}{100}cn$

$\Theta(1)$

$O(n)$ leaves

$\Theta(1)$

$\log_{10/9}n$

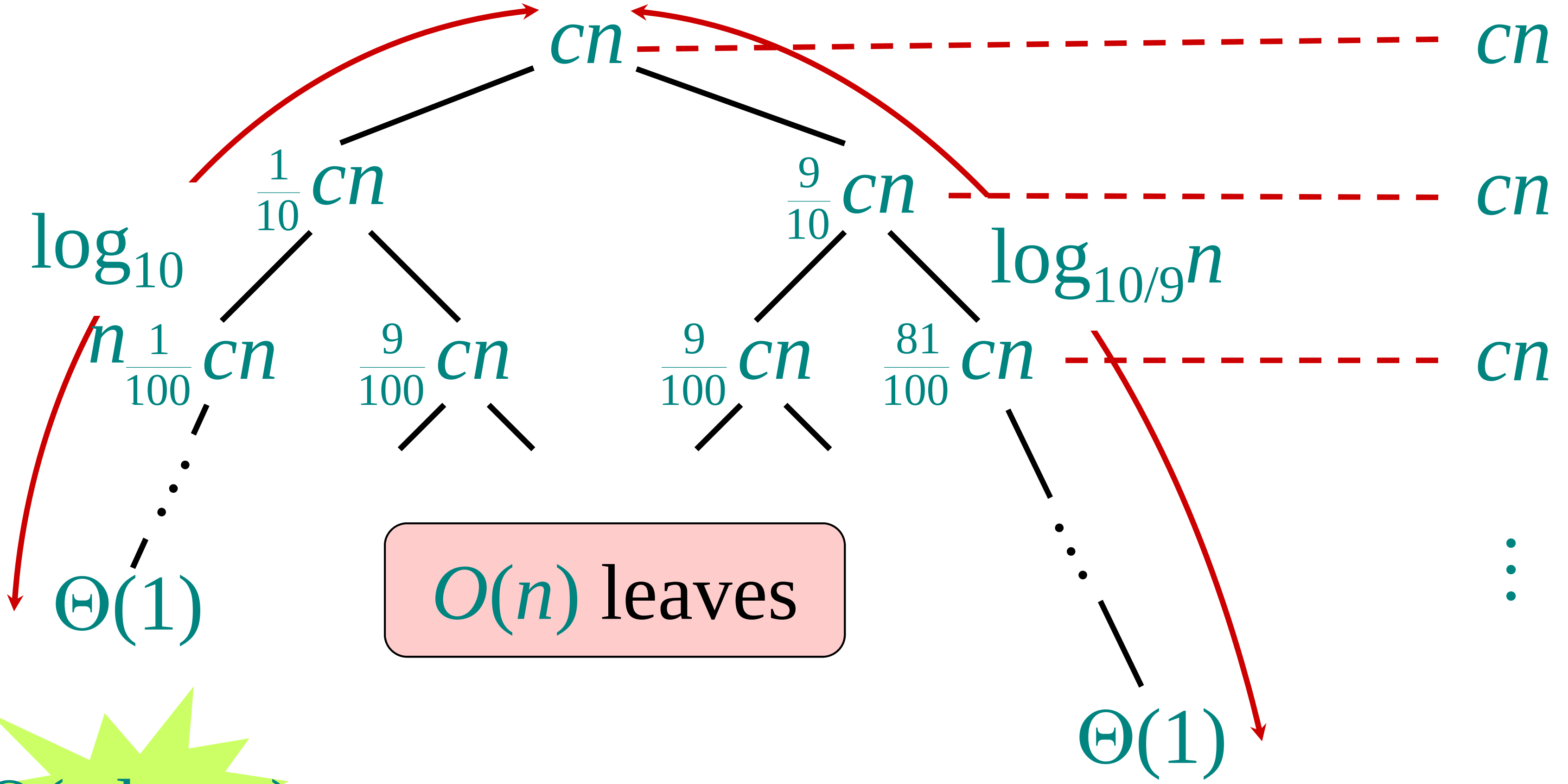
cn

cn

cn

$\vdots$

Analysis of “almost-best” case



cn

$\frac{1}{10}cn$ $\frac{9}{10}cn$

$\frac{1}{100}cn$ $\frac{9}{100}cn$ $\frac{9}{100}cn$ $\frac{81}{100}cn$

$\log_{10} n$

$\log_{10/9} n$

$\Theta(1)$

$\Theta(1)$

$O(n)$ leaves

cn

cn

cn

$\vdots$

$\Theta(n \log n)$

$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

Quicksort Runtimes

- Best case runtime $T_{\text{best}}(n) \in O(n \log n)$
- Worst case runtime $T_{\text{worst}}(n) \in O(n^2)$

- Worse than mergesort? Why is it called quicksort then?
- Its average runtime $T_{\text{avg}}(n) \in O(n \log n)$
- Better even, the expected runtime of **randomized quicksort** is $O(n \log n)$

Average Runtime

The **average runtime** $T_{avg}(n)$ for Quicksort is the average runtime over **all possible inputs** of length n.

- $T_{avg}(n)$ has to average the runtimes over all $n!$ different input permutations.
- There are still worst-case inputs that will have a $O(n^2)$ runtime

⇒ **Better:** Use randomized quicksort

Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order. It depends only on the sequence s of random numbers.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the sequence s of random numbers.

Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.

Average Runtime vs. Expected Runtime

- Average runtime is averaged over all inputs of a deterministic algorithm.
- Expected runtime is the expected value of the runtime random variable of a randomized algorithm. It effectively "averages" over all sequences of random numbers.

- De facto both analyses are very similar. However in practice the randomized algorithm ensures that not one single input elicits worst case behavior.

Order statistics

Select the ith smallest of n elements (the element with ***rank*** i).

- $i = 1$: ***minimum***;
- $i = n$: ***maximum***;
- $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: ***median***.

Naive algorithm: Sort and index ith element.

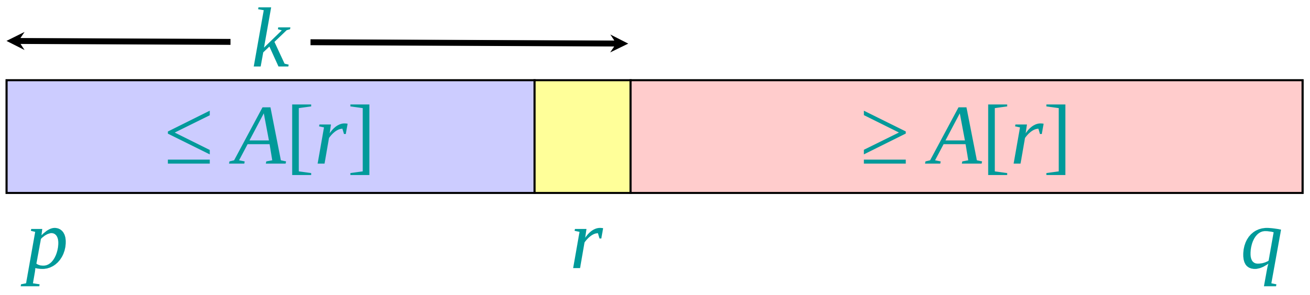
Worst-case running time $= \Theta(n \log n + 1)$

$= \Theta(n \log n)$,

using merge sort (*not* quicksort).

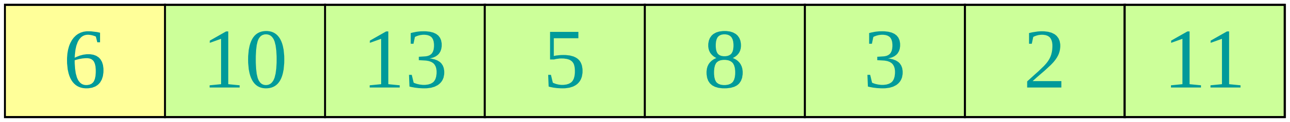
Randomized divide-and-conquer algorithm

```
RAND-SELECT($A, p, q, i$)          ⊳ $i$-th smallest of $A[p..q]$
    if $p = q$ then return $A[p]$
    $r \leftarrow$ RAND-PARTITION($A, p, q$)
    $k \leftarrow r - p + 1$                  ⊳ $k = \text{rank}(A[r])$
    if $i = k$ then return $A[r]$
    if $i < k$
        then return RAND-SELECT($A, p, r - 1, i$)
        else return RAND-SELECT($A, r + 1, q, i - k$)
```



Example

Select the $i = 7$th smallest:

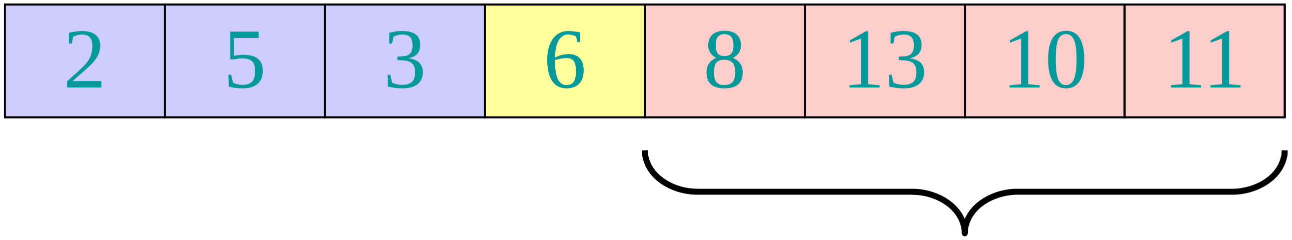


6	10	13	5	8	3	2	11

$i = 7$

pivot

Partition:



2	5	3	6	8	13	10	11

$k = 4$

Select the $7 - 4 = 3$rd smallest recursively.

Intuition for analysis

(All our analyses today assume that all elements are distinct.)

for RAND-PARTITION

Lucky:

$$T(n) = T(3n/4) + dn$$
$$= \Theta(n)$$

$$n^{\log_{4/3} 1} = n^0 = 1$$

CASE 3

Unlucky:

$$T(n) = T(n-1) + dn$$
$$= \Theta(n^2)$$

arithmetic series

Worse than sorting!

Analysis of expected time

- Call a pivot ***good*** if its rank lies in $[n/4, 3n/4]$.
- How many good pivots are there? $n/2$

$\Rightarrow$ A random pivot has 50% chance of being good.

- Let $T(n,s)$ be the runtime random variable

$$T(n,s) \le \overbrace{T(3n/4,s)}^{\text{time to reduce array size to } \le 3/4n} + \underbrace{\mathrm{X}(\mathrm{s})}_{\text{\#times it takes to find a good pivot}} \cdot \underbrace{dn}_{\text{Runtime of partition}}$$

Analysis of expected time

Lemma: A fair coin needs to be tossed an expected number of 2 times until the first “heads” is seen.

Proof: Let $E(X)$ be the expected number of tosses until the first “heads”is seen.

- Need at least one toss, if it’s “heads” we are done.
- If it’s “tails” we need to repeat (probability ½).

$$\Rightarrow E(X) = 1 + \tfrac{1}{2} E(X)$$

$$\Rightarrow E(X) = 2 \qquad \square$$

Analysis of expected time

time to reduce array size to $\leq 3/4n$

$$T(n,s) \leq T(3n/4,s) + \mathrm{X}(\mathrm{s})\cdot dn$$

#times it takes to find a good pivot

Runtime of partition

$$\Rightarrow E(T(n,s)) \leq E(T(3n/4,s)) + E(\mathrm{X}(\mathrm{s})\cdot dn)$$

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Linearity of expectation

$$\Rightarrow E(T(n,s)) \leq E(T(3n/4,s)) + 2\cdot dn$$

Lemma

$$\Rightarrow T_{exp}(n) \leq T_{exp}(3n/4) + \Theta(n)$$

$$\Rightarrow T_{exp}(n) \in \Theta(n)$$

□

Summary of randomized order-statistic selection

- Works fast: linear expected time.
- Excellent algorithm in practice.
- But, the worst case is ***very*** bad: $\Theta(n^2)$.

Q. Is there an algorithm that runs in linear time in the worst case?

A. Yes, due to Blum, Floyd, Pratt, Rivest, and Tarjan [1973].

IDEA: Generate a good pivot recursively. This algorithms large constants though and therefore is not efficient in practice.