More on Shortest Paths

Carola Wenk

Slides courtesy of Charles Leiserson with changes by Carola Wenk
Negative-weight cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:

Bellman-Ford algorithm: Finds all shortest-path weights from a source $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.
Bellman-Ford algorithm

\[ d[s] \leftarrow 0 \]
\[ \text{for each } v \in V - \{s\} \text{ do } d[v] \leftarrow \infty \]  \quad \text{initialization}

\[ \text{for } i \leftarrow 1 \text{ to } |V| - 1 \text{ do} \]
\[ \text{for each edge } (u, v) \in E \text{ do} \]
\[ \text{if } d[v] > d[u] + w(u, v) \text{ then } \]
\[ d[v] \leftarrow d[u] + w(u, v) \]
\[ \pi[v] \leftarrow u \]

\[ \text{relaxation step} \]

\[ \text{for each edge } (u, v) \in E \]
\[ \text{do if } d[v] > d[u] + w(u, v) \text{ then report that a negative-weight cycle exists} \]

At the end, \( d[v] = \delta(s, v) \). Time = \( O(|V||E|) \).
Example of Bellman-Ford

Order of edges: (B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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</thead>
<tbody>
<tr>
<td>Initialization</td>
<td>0</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
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Graph:
- A to B: -1
- A to C: 4
- A to D: ∞
- B to A: 3
- B to C: 2
- B to D: ∞
- B to E: ∞
- C to B: 3
- C to D: 1
- C to E: ∞
- D to A: 5
- D to B: 2
- D to C: ∞
- D to E: −3
- E to B: ∞
Example of Bellman-Ford

Order of edges: (B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)
Example of Bellman-Ford

Order of edges: \((B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)\)

\[
\begin{array}{cccccc}
A & B & C & D & E \\
0 & \infty & \infty & \infty & \infty \\
0 & -1 & \infty & \infty & \infty \\
0 & -1 & 4 & \infty & \infty \\
\end{array}
\]
Example of Bellman-Ford

Order of edges: \((B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)\)

```
\begin{array}{c|ccccc}
  & A & B & C & D & E \\
\hline
A & 0 & \infty & \infty & \infty & \infty \\
B & 0 & -1 & \infty & \infty & \infty \\
C & 0 & -1 & 4 & \infty & \infty \\
D & 0 & -1 & 2 & \infty & \infty \\
E & \infty & \infty & \infty & \infty & \infty \\
\end{array}
```

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Example of Bellman-Ford

Order of edges: \((B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)\)
Example of Bellman-Ford

Order of edges: \((B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)\)

\[
\begin{array}{c|ccccc}
&A&B&C&D&E \\
\hline
0&\infty&\infty&\infty&\infty&\infty \\
0&-1&\infty&\infty&\infty&\infty \\
0&-1&4&\infty&\infty&\infty \\
0&-1&2&\infty&\infty&1 \\
\end{array}
\]
Example of Bellman-Ford

Order of edges: \((B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)\)

\[
\begin{array}{ccc|ccc}
A & B & C & D & E \\
\hline
0 & \infty & \infty & \infty & \infty \\
0 & -1 & \infty & \infty & \infty \\
0 & -1 & 4 & \infty & \infty \\
0 & -1 & 2 & \infty & \infty \\
0 & -1 & 2 & \infty & 1 \\
0 & -1 & 2 & 1 & 1 \\
\end{array}
\]
Example of Bellman-Ford

Order of edges: $(B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)$

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Example of Bellman-Ford

Order of edges: \((B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D)\)

\[\begin{array}{cccccc}
A & B & C & D & E \\
0 & \infty & \infty & \infty & \infty \\
0 & -1 & \infty & \infty & \infty \\
0 & -1 & 4 & \infty & \infty \\
0 & -1 & 2 & \infty & \infty \\
0 & -1 & 2 & 1 & 1 \\
0 & -1 & 2 & -2 & 1 \\
\end{array}\]

Note: Values decrease monotonically.

\[\ldots\text{ and 2 more iterations}\]
Correctness

**Theorem.** If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

**Proof.** Let $v \in V$ be any vertex, and consider a shortest path $p$ from $s$ to $v$ with the minimum number of edges.

Since $p$ is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i).$$
Correctness (continued)

Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[s]$ is unchanged by subsequent relaxations.

• After 1 pass through $E$, we have $d[v_1] = \delta(s, v_1)$.
• After 2 passes through $E$, we have $d[v_2] = \delta(s, v_2)$.
...
• After $k$ passes through $E$, we have $d[v_k] = \delta(s, v_k)$.

Since $G$ contains no negative-weight cycles, $p$ is simple. Longest simple path has $\leq |V| - 1$ edges.
Detection of negative-weight cycles

Corollary. If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in $G$ reachable from $s$. □
DAG shortest paths

If the graph is a *directed acyclic graph* (*DAG*), we first *topologically sort* the vertices.

- Determine $f : V \rightarrow \{1, 2, \ldots, |V|\}$ such that $(u, v) \in E \Rightarrow f(u) < f(v)$. 

![Diagram of a DAG with vertices labeled 1 to 9 and directed edges between them.](image)
DAG shortest paths

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- Determine \( f : V \rightarrow \{1, 2, \ldots, |V|\} \) such that \((u, v) \in E \Rightarrow f(u) < f(v)\).
- \( O(|V| + |E|) \) time

- Walk through the vertices \( u \in V \) in this order, relaxing the edges in \( \text{Adj}[u] \), thereby obtaining the shortest paths from \( s \) in a total of \( O(|V| + |E|) \) time.
Shortest paths

Single-source shortest paths
• Nonnegative edge weights
  • Dijkstra’s algorithm: $O(|E| \log |V|)$
• General: Bellman-Ford: $O(|V||E|)$
• DAG: One pass of Bellman-Ford: $O(|V| + |E|)$
Shortest paths

**Single-source shortest paths**
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**All-pairs shortest paths**
- Nonnegative edge weights
  - Dijkstra’s algorithm $|V|$ times: $O(|V||E| \log |V|)$
  - General
    - Bellman-Ford $|V|$ times: $O(|V|^2|E|)$
    - Floyd-Warshall: $O(|V|^3)$