CMPS/MATH 2170 Discrete Mathematics – Fall 15

10/7/15

6. Homework

Due **10/19/15** at the beginning of class

1. **Weak induction in steps (7 points)**
   Let $P(n)$ be the following statement:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

   The following sub-problems guide you through a proof by weak induction that $P(n)$ holds for all $n \in \mathbb{N}$.

   (a) (1 point) In order to understand the claim, verify it by hand for $n = 3$.

   (b) (1 point) What is the statement $P(1)$? Show that $P(1)$ is true, which completes the base case.

   (c) (1 point) What is the inductive hypothesis?

   (d) (1 point) What do you need to prove in the inductive step?

   (e) (3 points) Complete the inductive step.

2. **More weak induction (10 points)**
   Use weak induction on n to prove the following claims.

   (a) (5 points) $3^n < n!$ for all integers $n \geq 7$.

   (b) (5 points) Let $n \in \mathbb{N}$ and let $A_1, A_2, \ldots, A_n$ and B be sets. Prove that $(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \cup B)$.

3. **Strong induction (5 points)**
   Prove the following claim using strong induction:

   Every positive integer n can be written as the sum of distinct powers of two.

   Hint: (For example, $11 = 8 + 2 + 1 = 2^3 + 2^1 + 2^0$. In your proof, distinguish whether n is even or odd.)

4. **Fibonacci (5 points)**
   Let f_i be the i-th Fibonacci number and let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Use induction to show that

$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

   for any positive integer n.