# CMPS/MATH 2170 – Fall 2013

# *Graphs*

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# Graphs (review)

**Definition.** A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set $V$ of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set $E$ consists of *unordered* pairs of vertices.

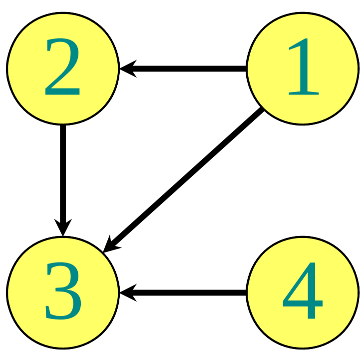
In either case, we have $|E| = O(|V|^2)$.

Moreover, if $G$ is connected, then $|E| \geq |V| - 1$.

# Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1\,.\,.\,n, 1\,.\,.\,n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$

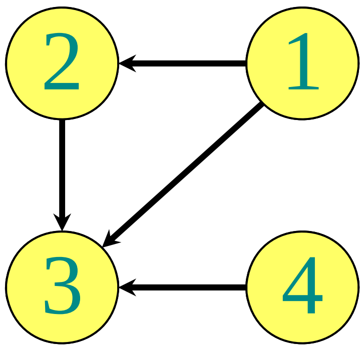



| $A$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

$\Theta(|V|^2)$ storage $\Rightarrow$ ***dense*** representation.

# Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to $v$.




$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.

For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

# Adjacency-list representation

**Handshaking Lemma:**

*Every edge is counted twice*

- For undirected graphs:

  $\sum_{v \in V} \mathit{degree}(v) = 2|E|$

- For digraphs:

  $\sum_{v \in V} \mathit{in\text{-}degree}(v) + \sum_{v \in V} \mathit{out\text{-}degree}(v) = 2\,|\,E\,|$

$\Rightarrow$ adjacency lists use $\Theta(|V| + |E|)$ storage

$\Rightarrow$ a ***sparse*** representation

# Paths, Cycles, Connectivity

Let $G=(V,E)$ be a directed (or undirected) graph

- A **path** from $v_1$ to $v_k$ in $G$ is a sequence of vertices $v_1, v_2,\ldots,v_k$ such that $(v_i,v_{\{i+1\}})\in E$ (or $\{v_i,v_{\{i+1\}}\} \in E$ if $G$ is undirected) for all $i\in\{1,\ldots,k\text{-}1\}$.
- A path is **simple** if all vertices in the path are distinct.
- A path $v_1, v_2,\ldots,v_k$ forms a **cycle** if $v_1=v_k$ .
- A graph with no cycles is **acyclic**.
  - An undirected acyclic connected graph is called a **tree**. (Trees do not have to have a root vertex specified.)
  - A directed acyclic graph is a **DAG**. (A DAG can have undirected cycles if the direction of the edges is not considered.)
- An undirected graph is **connected** if every pair of vertices is connected by a path. A directed graph is **strongly connected** if for every pair $u,v\in V$ there is a path from $u$ to $v$ and there is a path from $v$ to $u$.
- The **(strongly) connected components** of a graph are the equivalence classes of vertices under this reachability relation.