On Minimum Area Homotopies

Brittany Terese Fasy, Selcuk Karakoc, Carola Wenk

MONTANA STATE UNIVERSITY

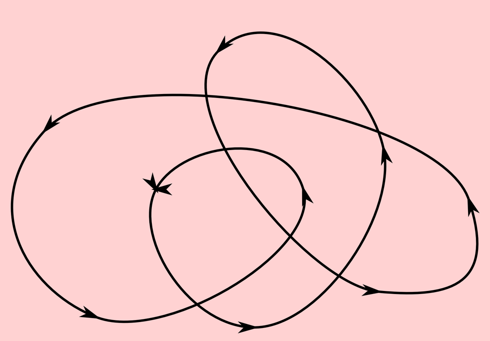
Tulane University

Introduction

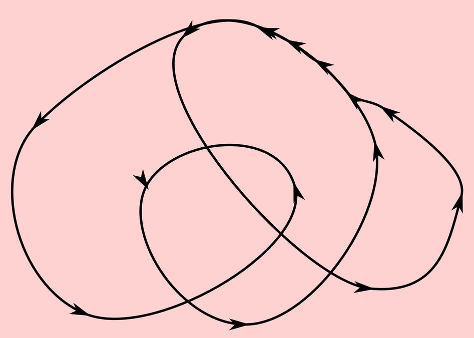
- Minimum homotopy area between two simple curves has been defined by Chambers and Wang [1]. Here, we generalize it for closed curves and we give a method to compute it. Our method consists of the following two steps:
- We compute the minimum homotopy area for a class of closed curves, namely self-overlapping curves.
- We show that any closed curve can be devided into self-overlapping subcurves in such a way that the minimum homotopy area of the curve is the sum of the minimum homotopy areas of self-overlapping subcurves.

Normal Curves and Titus Moves

- A piecewise regular closed curve is a piecewise differentiable map $C : [0,1] \to \mathbb{R}^2$ such that $C(0) = C(1)$ and the derivative C' never vanishes whenever it is defined. We denote $[C]$ for the image of the map.
- A point $p \in [C]$ is called ordinary if the preimage $C^{-1}(p)$ consists of one point. A point $p \in [C]$ is called a simple crossing point if there exist exactly two points $t_1, t_2 \in [0,1]$ such that $p = C(t_1) = C(t_2)$ and if $C'(t_1)$ and $C'(t_2)$ are linearly independent. A piecewise regular closed curve C is called normal if there exist only a finite number of simple crossing points and all other points of $[C]$ are ordinary.

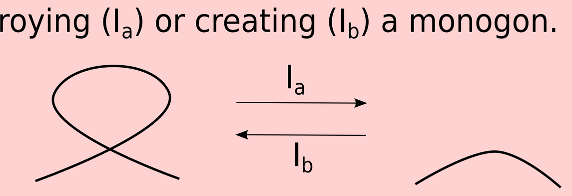


A normal curve.

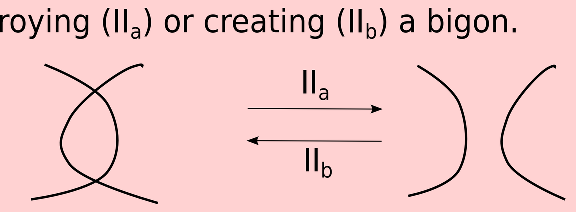


A curve which is not normal.

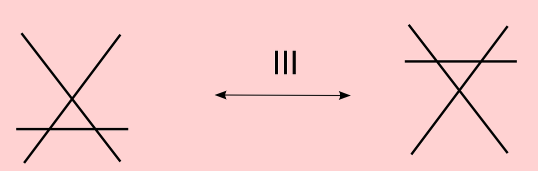
- A homotopy H between two normal curves C_1 and C_2 sharing same end points can be considered as a sequence of three different moves, which we call Titus moves:
  - Type 1: Destroying (I_a) or creating (I_b) a monogon.



  - Type 2: Destroying (II_a) or creating (II_b) a bigon.



  - Type 3: Inverting a triangle.

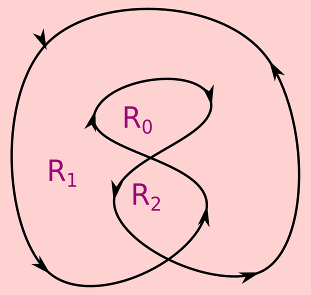


- Since $\mathbb{R}^2$ is simply connected, any two closed curves with the same end points are homotopic to each other. In particular, any closed curve C is homotopic to the constant path p_0 where $p_0(t) = C(0) \text{ for all } t \in [0,1]$. A homotopy H between C_1 and C_2 is denoted by $C_1 \xrightarrow{H} C_2$. A I_a move takes a simple sub-loop of an intermediate curve and contracts it to a point in the plane. We call such points the anchor points of the homotopy.

Homotopy Area and Winding Area of a Closed Curve

- Let $wn(x,C)$ be the winding number of the curve C at a point x in the plane. We define the winding area $W(C)$ of C as follows:

$$W(C) = \int_{\mathbb{R}^2} |wn(x,C)| dx$$



Consider the curve on the left. We have:
- $wn(x,C) = 0 \text{ for } x \in R_0$
- $wn(x,C) = 1 \text{ for } x \in R_1$
- $wn(x,C) = 2 \text{ for } x \in R_2$

Hence, $W(C) = 2\,Area(R_2) + Area(R_1)$

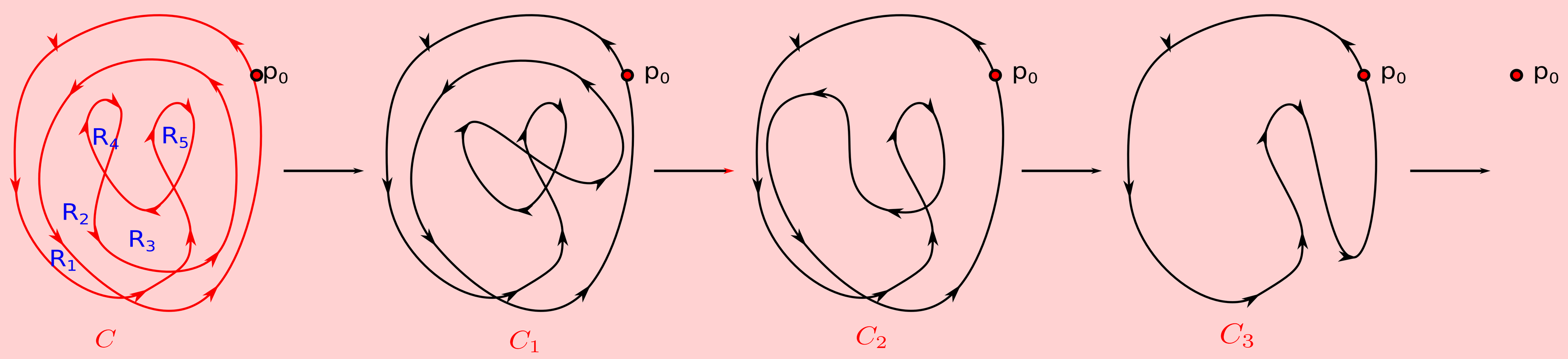
- Let $C_1 \xrightarrow{H} C_2$ be a homotopy and $E_H(x)$ be the number of connected components of $H^{-1}(x)$ for a point x in the plane. We define the area of the homotopy as follows:

$$Area(H) = \int_{\mathbb{R}^2} E_H(x)\; dx$$

We define the minimum homotopy area of C_1 and C_2 as the infimum of the areas over all possible homotopies between them. We denote it by $\sigma(C_1, C_2)$. In other words:

$$\sigma(C_1, C_2) = \inf_H Area(H)$$

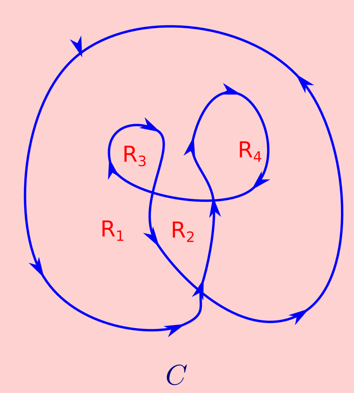
We also define $\sigma(C) = \sigma(C, p_0)$. A homotopy that realizes the above infimum is called a minimum homotopy.



The sequence of Titus moves in this figure comprises a minimum homotopy of C. We refer to this curve as the red curve. For the red curve, we have;

$$\sigma(C) = W(C) = 3\,Area(R_3) + 2\;Area(R_2) + Area(R_1) + Area(R_4) + Area(R_5)$$

- For some curves, homotopy area and the winding area are equal; see the red curve above.



We refer the curve on the left as the blue curve. For the blue curve, we have;

$$\sigma(C) = 2Area(R_2) + 2Area(R_3) + Area(R_1) > W(C) = \; 2\;Area(R_2) + Area(R_1)$$

For an arbitrary curve, we have the following lemma:

Lemma: For any normal curve, we have $\sigma(C) > W(C)$.

- Notice that minimum homotopy area between two curves C_1 and C_2 is zero if and only if they have the same image, i.e. $[C_1] = [C_2]$. Obviously, it is symmetric, $\sigma(C_1, C_2) = \sigma(C_2, C_1)$. It also satisfies the triangular inequality. Hence, the minimum homotopy area defines a metric on the space of normal curves with a fixed end point under the equivalence realtion where two curves are equal if and only if they have the same image.

Self-Overlapping Curves and the Main Theorem

- A normal curve C is called self-overlapping if there exists an immersion of the disk $F : D^2 \to \mathbb{R}^2$ such that $F|_{\partial D^2} = [C]$. The red curve is an example of a self-overlapping curve, whereas the blue curve is an example of a non-self-overlapping curve. Self-overlapping curves have consistent winding numbers. In other words, winding numbers are all non-negative or all non-positive for each point in the plane. Furthermore, we have the following theorem:

  Theorem: If C is self-overlapping, then $\sigma(C) = W(C)$.

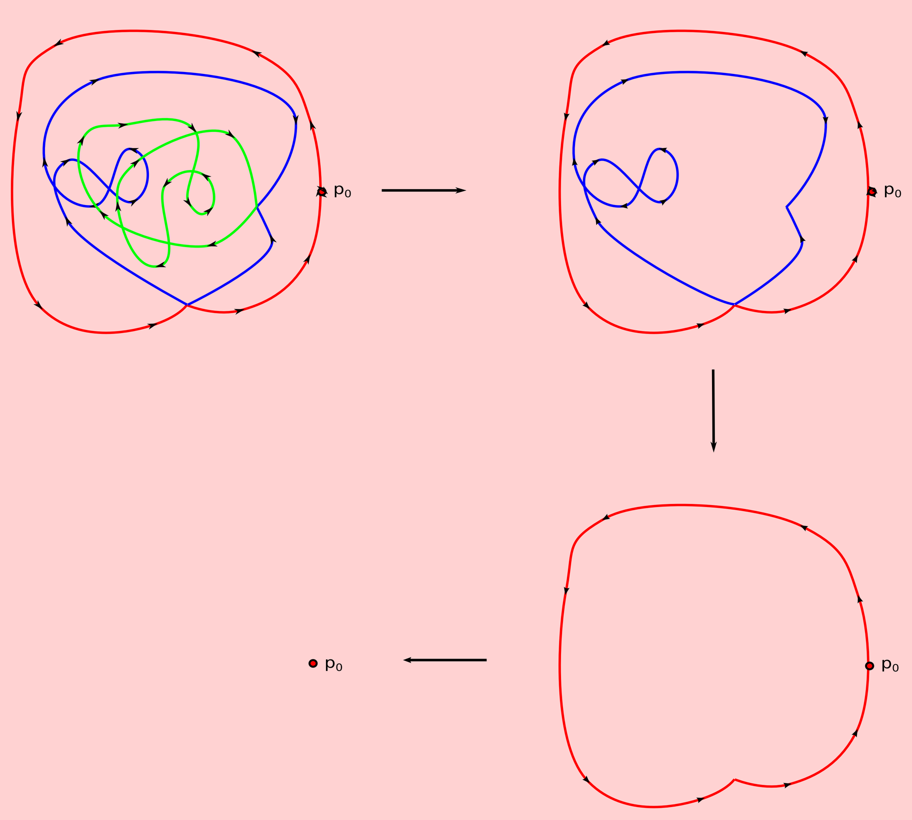
  Detecting whether a given curve is self-overlapping or not can be done in polynomial time [2].

- Now, we state our main theorem.

  Theorem: Let C be a normal curve. Then, there exists a minimum homotopy H which defines a sequence of curves $C = C_0 \to C_1 \to \ldots \to C_m = p_0$ such that each $C_i \to C_{i+1}$ is a contraction of a self-overlapping subcurve of C_i based at a simple crossing point of C_i.

  Proof (Sketch):
  We show that for each normal curve there exists a minimum area homotopy that does not require I_b moves, and any II_b move does not create anchor points. Furthermore, if these homotopies are carefully constructed, the anchor points will be a subset of the simple crossing points of the curve. And since a minimum homotopy is locally sense-preserving, these anchor points define self-overlapping pieces of the curve. □

- In the figure below, we demonstrate our theorem by first subdividing a curve into three self-overlapping sub-curves (top left). This decomposition is not unique, but one such subdivision will realize the minimum homotopy. We illustrate one minimum homotopy.



- Testing all possible sequences of intersection points as anchor points yields an exponential-time algorithm. We are currently working on using dynamic programming to obtain a polynomial-time algorithm.

References:
[1] Chambers, E., and Wang, Y. Measuring similarity between curves on 2-manifolds via homotopy area. In SoCG (2013), pp. 425-434.
[2] Shor, P., and Van Wyk, C. Detecting and decomposing self-overlapping curves. CGTA 2, 1 (1992), 31-50.

Research reported in this poster was supported by NSF grant CCF-1301911.