On Minimum Area Homotopies
Brittany Terese Fasy, Selcuk Karakoc, Carola Wenk

Introduction

- Minimum homotopy area between two simple curves has been defined by Chambers and Wang [1]. Here, we generalize it for closed curves and we give a method to compute it. Our method consists of the following two steps:
  - We compute the minimum homotopy area for a class of closed curves, namely self-overlapping curves.
  - We show that any closed curve can be divided into self-overlapping subcurves in such a way that the minimum homotopy area of the curve is the sum of the minimum homotopy areas of self-overlapping subcurves.

Normal Curves and Titus Moves

- A piecewise regular closed curve is a piecewise differentiable map \( C : [0, 1] \to \mathbb{R}^2 \) such that \( C(0) = C(1) \) and the derivative \( C' \) never vanishes whenever it is defined. We denote \( C' \) for the image of the map.
- A point \( x \in \mathbb{R}^2 \) is called ordinary if the preimage \( x^{-1}(x) \) consists of one point. A point \( x \in \mathbb{R}^2 \) is called a simple crossing point if there exist exactly two points \( t, s, t < s \) such that \( x = C(t) = C(s) \) and \( C'(t), C'(s) \) are linearly independent. A piecewise regular closed curve \( C \) is called normal if there exist only a finite number of simple crossing points and all other points of \( C \) are ordinary.

Homotopy Area and Winding Area of a Closed Curve

- Let \( \nu_C(x) \) be the winding number of the curve \( C \) at a point \( x \) in the plane. We define the winding area \( W(C) \) of \( C \) as follows:

\[
W(C) = \int_{x \in \mathbb{R}^2} \nu_C(x) \, dx
\]

Consider the curve on the left. We have:
- \( \nu_C(x) = 0 \) for \( x \in R_3 \)
- \( \nu_C(x) = 1 \) for \( x \in R_1 \)
- \( \nu_C(x) = 2 \) for \( x \in R_2 \)

Hence, \( W(C) = 2 \cdot \text{Area}(R_2) - \text{Area}(R_1) \)

- Let \( G_1, G_2 \) be a homotopy and \( E_G(x) \) be the number of connected components of \( G \), then we have:

\[
\text{Area}(H) = \int_{x \in \mathbb{R}^2} E_G(x) \, dx
\]

We define the minimum homotopy area of \( G_1 \) and \( G_2 \) as the infimum of the areas over all possible homotopies as follows:

\[
\text{Area}(H) = \inf \left\{ \int_{x \in \mathbb{R}^2} E_G(x) \, dx \right\}
\]

We also define \( r(C) = r(C, p_0) \). A homotopy that realizes the above infimum is called a minimum homotopy.

- The sequence of Titus moves in this figure comprises a minimum homotopy of \( C \). We refer to this curve as the red curve. For the red curve, we have:

\[
r(C) = W(C) = 3 \cdot \text{Area}(R_3) + 2 \cdot \text{Area}(R_2) + \text{Area}(R_1) + \text{Area}(R_0)
\]

- For some curves, homotopy area and the winding area are equal; see the red curve above.

\[
r(C) = W(C) = 3 \cdot \text{Area}(R_3) + 2 \cdot \text{Area}(R_2) + \text{Area}(R_0) > W(C) = 2 \cdot \text{Area}(R_2) + \text{Area}(R_1)
\]

For an arbitrary curve, we have the following lemma:

Lemma: For any normal curve, we have \( r(C) \geq W(C) \).

Self-Overlapping Curves and the Main Theorem

- A normal curve \( C \) is called self-overlapping if there exists an immersion of the disk \( D^2 \) into \( \mathbb{R}^2 \) such that \( \gamma(D^2) = \{ C \} \). The red curve is an example of a self-overlapping curve, whereas the blue curve is an example of a non-self-overlapping curve. Self-overlapping curves have consistent winding numbers. In other words, winding numbers are all non-negative or all non-positive for each point in the plane. Furthermore, we have the following theorem:

Theorem: If \( C \) is self-overlapping, then \( r(C) = W(C) \).

- Detecting whether a given curve is self-overlapping or not can be done in polynomial time [2].

- Now, we state our main theorem.

Theorem: Let \( C \) be a normal curve. Then, there exists a minimum homotopy \( H \) which defines a sequence of curves \( C_0 = C_1 = \ldots = C_n = p_0 \) such that each \( C_i \to C_{i+1} \) is in a contraction of a self-overlapping subcurve of \( C_i \) based at a simple crossing point of \( C_i \).

Proof (Sketch): We show that for each normal curve there exists a minimum area homotopy that does not require \( I_3 \) moves, and any \( I_4 \) move does not create anchor points. Furthermore, if these homotopies are carefully constructed, the anchor points will be a subset of the simple crossing points of the curve. And since a minimum homotopy is locally sense-preserving, these anchor points define self-overlapping pieces of the curve.

- In the figure below, we demonstrate our theorem by first subdividing a curve into three self-overlapping sub-curves (top left). This decomposition is not unique, but one such subdivision will realize the minimum homotopy. We illustrate one minimum homotopy.

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References: