Induction and Recursion

CMPS/MATH 2170: Discrete Mathematics

Outline

- Mathematical induction (5.1)
- Sequences and Summations (2.4)
- Strong induction (5.2)
- Recursive definitions (5.3)
- Recurrence Relations (8.1)

Principle of Mathematical Induction

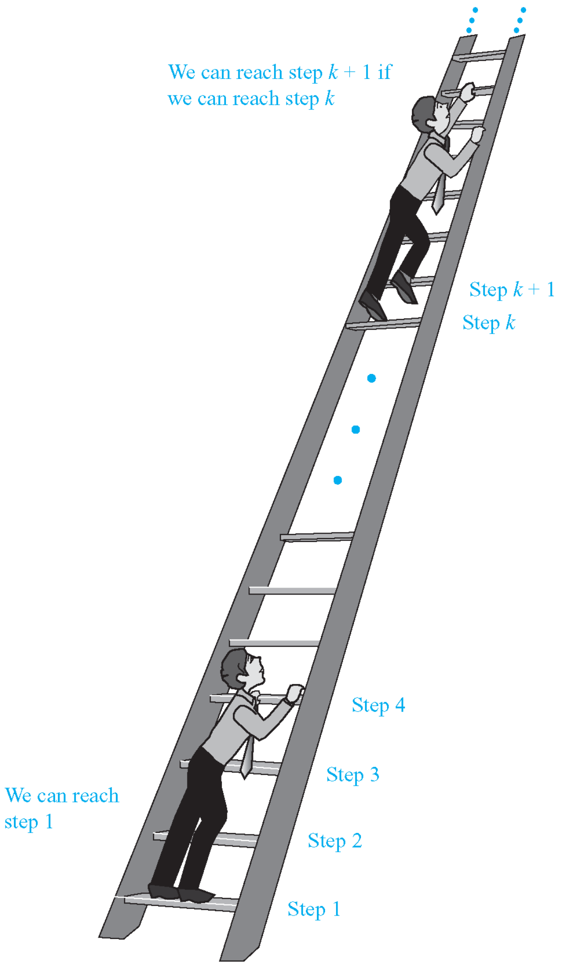

FIGURE 1 **Climbing an Infinite Ladder.**

- Want to know if we can reach every step on a infinite ladder
- Suppose we know two things
  - We can reach the first rung of the ladder
  - If we can reach a particular rung of the ladder, then we can reach the next rung
- Can we conclude that we can reach every rung?

Mathematical Induction

- Want to show: $\forall n \in \mathbb{Z}^+ : P(n)$

  Proof by induction on n

  - Base case: verify that $P(1)$ is true
  - Inductive step: show that $P(k) \rightarrow P(k+1)$ for any $k \in \mathbb{Z}^+$

    Inductive hypothesis: Assume $P(k)$ is true

    Want to prove $P(k+1)$ is true

Examples

Ex. 1: Prove that $\underbrace{3 \text{ divides } n^3 - n}_{P(n)}$ for any $n \in \mathbb{Z}^+$

Mathematical Induction

- Want to show: ~~$\forall n \in \mathbb{Z}^+$~~: $P(n)$ for $n = b, b+1, b+2, \dots,$ where $b \in \mathbb{Z}$
- Proof by induction on n
  - Base case: verify that ~~$P(1)$~~ $P(b)$ is true
  - Inductive step: show that $P(k) \rightarrow P(k+1)$ for any ~~$k \in \mathbb{Z}^+$~~ $k \in \mathbb{Z}$ and $k \geq b$

    Inductive hypothesis: Assume $P(k)$ is true

    Want to prove $P(k+1)$ is true

Examples

Ex. 1: Prove that 3 divides $n^3 - n$ for any $n \in \mathbb{Z}^+$

Ex. 2: Prove $n^2 < 2^n$ for all integers $n > 4$

Ex. 3: Prove that a finite set with n elements has 2^n subsets

Ex. 4: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

More about Mathematical Induction

- Why Mathematical Induction is valid?
  - Implied by the Weak Ordering Property:

    "Every nonempty subset of $\mathbb{Z}^+$ has a least element"

$$\begin{array}{l} P(1) \\ \forall k \in \mathbb{Z}^+ \colon P(k) \rightarrow P(k+1) \\ \hline \therefore \forall n \in \mathbb{Z}^+ \colon P(n) \end{array}$$

- Pros
  - Can be used to prove a wide variety of "forall" conjectures
  - Easy to follow
- Cons
  - Cannot be used to find new theorems
  - Lack of insight

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Sequences

- Informally, a sequence is an ordered list of objects
  - List all positive even integers: 2, 4, 6, 8, 10, …
  - We can describe the sequence as $\{a_i\}_{i \in \mathbb{Z}^+}$ where $a_i = 2i$
- Formally, a sequence is a function with domain $\mathbb{Z}^+$ or $\mathbb{N}$:
  - the above sequene can be defined by

$$f:\ \mathbb{Z}^+ \to \mathbb{Z}^+$$

$$i \longmapsto 2i$$

Arithmetic Progression

- Consider a sequence: 1, 4, 7, 10, 13, 16, 19…
  - We can represent it as $a_i = 1 + 3 \cdot i, \; i \geq 0$

- An arithmetic progression is a sequence $\{a_i\}_{i \in \mathbb{N}}$ with $a_i = b + c \cdot i$, for $b, c \in \mathbb{R}$

$$a_0 = b,$$
$$a_1 = b + c,$$
$$a_2 = b + 2c,$$
$$\ldots$$

$$f\colon \; \mathbb{N} \to \mathbb{R}$$
$$i \longmapsto b + ci$$

Geometric Progression

- Consider a sequence: 3, 6, 12, 24, 48, 96, …
  - We can represent it as $a_i = 3 \cdot 2^i, \quad i \geq 0$
- A geometric progression is a sequence $\{a_i\}_{i \in \mathbb{N}}$ with $a_i = b \cdot c^i$, for $b, c \in \mathbb{R}$

$$a_0 = b,$$
$$a_1 = bc,$$
$$a_2 = bc^2,$$
$$\ldots$$

$$f\colon\ \mathbb{N} \to \mathbb{R}$$
$$i \longmapsto bc^i$$

Summations

For a sequence $\{a_i\}$, we write

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

Summations

Ex.1: Sums of Arithmetic Progressions

$$\forall n \in \mathbb{N}: \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Ex.2: Sums of Geometric Progressions

$$\forall n \in \mathbb{N}: \sum_{i=0}^{n} c^i = \frac{c^{n+1} - 1}{c - 1} \qquad \text{where } c \neq 1$$

Proof by Mathematical Induction

Summations

$$\sum_{i=5}^{10} 2^i = \sum_{i=0}^{10} 2^i - \sum_{i=0}^{4} 2^i$$

$$\sum_{i=5}^{10} 2^i = \sum_{j=0}^{5} 2^{j+5} = 32 \times \sum_{j=0}^{5} 2^j$$

Index substitution: $j = i - 5$

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Strong Induction

- Want to prove: $\forall n \in \mathbb{Z}^+ \colon P(n)$

Proof by (weak) induction on n:

- Base case: verify that $P(1)$ is true
- Inductive step: show that $P(k) \rightarrow P(k+1)$ for any $k \in \mathbb{Z}^+$

Proof by strong induction on n:

- Base case: verify that $P(1)$ is true
- Inductive step: show that $[P(1) \wedge P(2) \wedge \ \ldots \wedge P(k)] \rightarrow P(k+1)$ for any $k \in \mathbb{Z}^+$

Strong Induction

- A more general form of strong induction
  - Want to prove: $P(n)$ for $n = b, b+1, b+2, \dots,$ where $b \in \mathbb{Z}$
  - Base step: verify that $P(b), P(b+1), \dots P(b+j)$ are true
  - Inductive step: Assume $[P(b) \wedge P(b+1) \wedge \ \dots \wedge P(k)]$ is true, prove $P(k+1)$ is true for every integer $k \geq b+j$

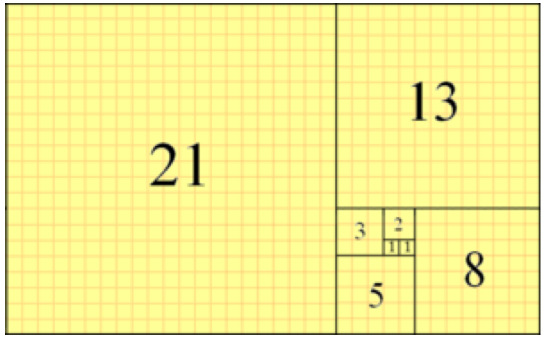
Examples of Strong Induction

Fibonacci Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, …

$f_0 = 0,\ f_1 = 1,$ Initial conditions

$f_n = f_{n-1} + f_{n-2}, n \geq 2$ Recurrence relation

This is called a recursive definition



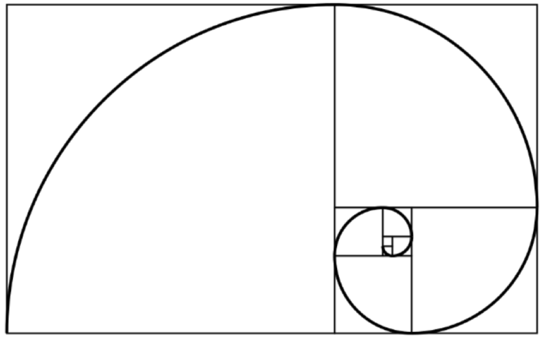
Fibonacci Tiling

Ex.1: $f_n \leq 2^n$ for all $n \geq 0$

Ex. 2: $f_n > \alpha^{n-2}$ for any $n \geq 3$ where $\alpha = (1 + \sqrt{5})/2$

≈ 1.618

(golden ratio)


Fibonacci Spiral

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Recursively Defined Sequences

- Fibonacci Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, …

  $f_0 = 0, \;\; f_1 = 1,$ Initial conditions

  $f_n = f_{n-1} + f_{n-2}, \;\; n \geq 2$ Recurrence relation

- A sequence of powers of 2: 1, 2, 4, 8, 16, 32 …

  An explicit formula: $a_n = 2^n, \; n \geq 0$

  A recursive definition:

  $a_0 = 2^0,$ Initial condition

  $a_n = 2a_{n-1}, \; n \geq 1$ Recurrence relation

Recursively Defined Functions

- A recursive definition of $f: \mathbb{N} \to \mathbb{R}$, $\mathbb{N} = \{0,1,2,3 \dots\}$
  - Base step: specify $f(0)$
  - Recursive step: specify $f(n)$ in terms of $f(0), f(1), \dots, f(n-1)$, for any $n \geq 1$

- Ex.1: Give a recursive definition of $f(n) = n!$

- Ex.2: Give a recursive definition of $f(n) = \sum_{i=0}^{n} a_i$, where $\{a_i\}_{i \in \mathbb{N}}$ is a given sequence

Recursively Defined Sets

- Consider a set $S \subseteq \mathbb{Z}^+$ recursively defined by
  - Base step: $3 \in S$
  - Recursive step: if $x \in S$ and $y \in S$, then $x + y \in S$
- S is the set of all positive integers divided by 3

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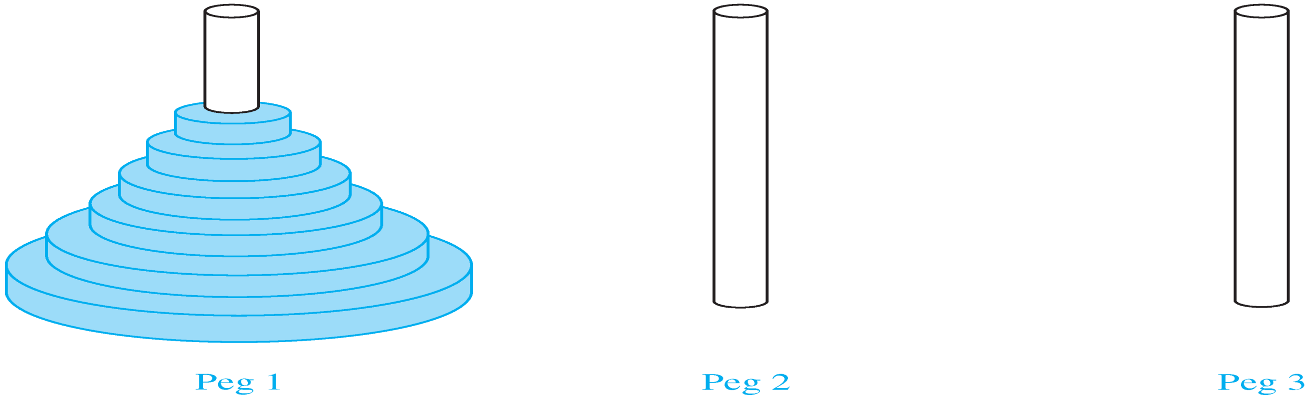
The Tower of Hanoi



Task: Move the stack of disks from peg 1 to peg 3 subject to the following rules:

- Move one disk at a time
- Only the uppermost disk on a stack can be moved
- No disk can be placed on top of a smaller disk

Question: how many steps are needed?

The Tower of Hanoi

By André Karwath aka Aka - Own work, CC BY-SA 2.5,
https://commons.wikimedia.org/w/index.php?curid=85401

The Tower of Hanoi

- H_n - number of moves needed to solve the Tower of Hanoi with n disks

- Divide & Conquer

  $H_1 = 1$ Initial condition

  $H_n = 2H_{n-1} + 1, n \geq 2$ Recurrence relation

- How to find an explicit formula for H_n?
  - Expand the recurrence iteratively and then make a conjecture: $H_n = 2^n - 1$
    - $2^{64} - 1$ seconds ≈ 585 billion years
  - Proof by mathematical induction